f4ncgb: High Performance Gröbner Basis Computations in Free Algebras

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JOHANNES KEPLER UNIVERSITY LINZ

FWF
Der Wissenschaftsfonds.

Noncommutative Polynomials

noncommutative = really noncommutative

Noncommutative Polynomials

$$\begin{aligned} \text{noncommutative} &= \text{really noncommutative} \\ &= \text{no commutation rules} \end{aligned}$$

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Noncom. polynomial $\quad c_1 \cdot w_1 + \cdots + c_d \cdot w_d \quad \in K\langle x_1, \ldots, x_n \rangle$

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words over $x_1, \ldots, x_n$

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$\in K$

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Example $\quad xyyx+2xy-yx-2 \in \mathbb{Q}\langle x,y\rangle$

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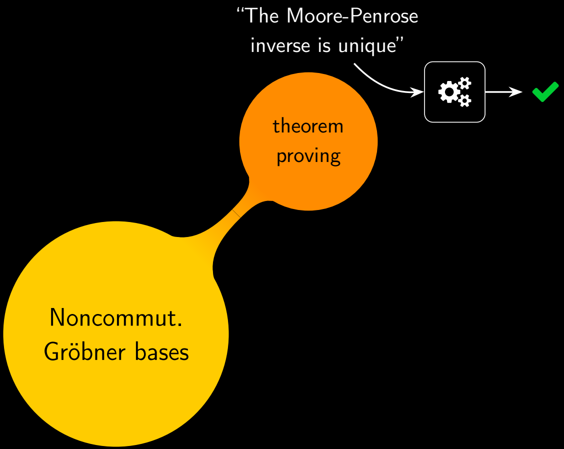
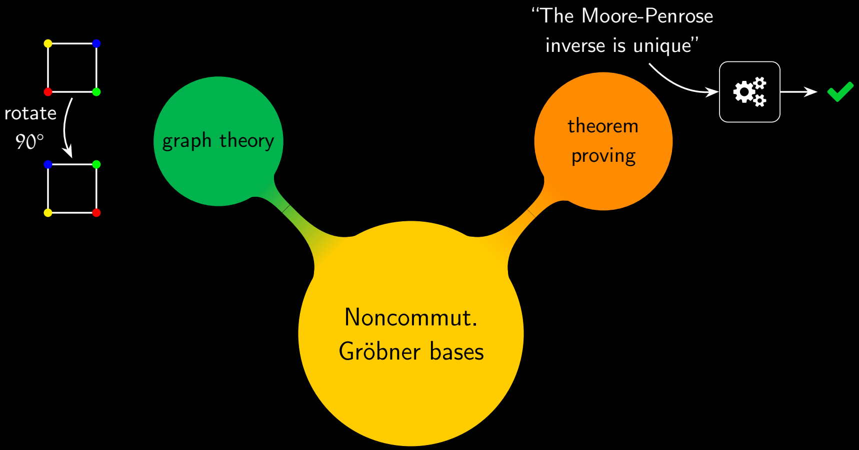
Noncomm. GB theory = comm. GB theory − finiteness

Noncommut.
Gröbner bases


"The Moore-Penrose
inverse is unique"
theorem
proving
Noncommut.
Gröbner bases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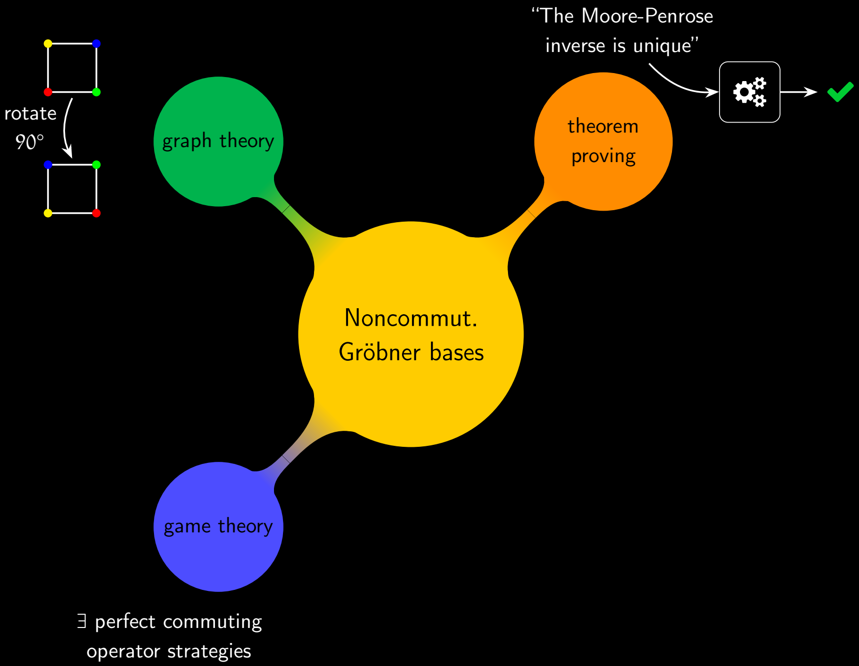
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∃ perfect commuting
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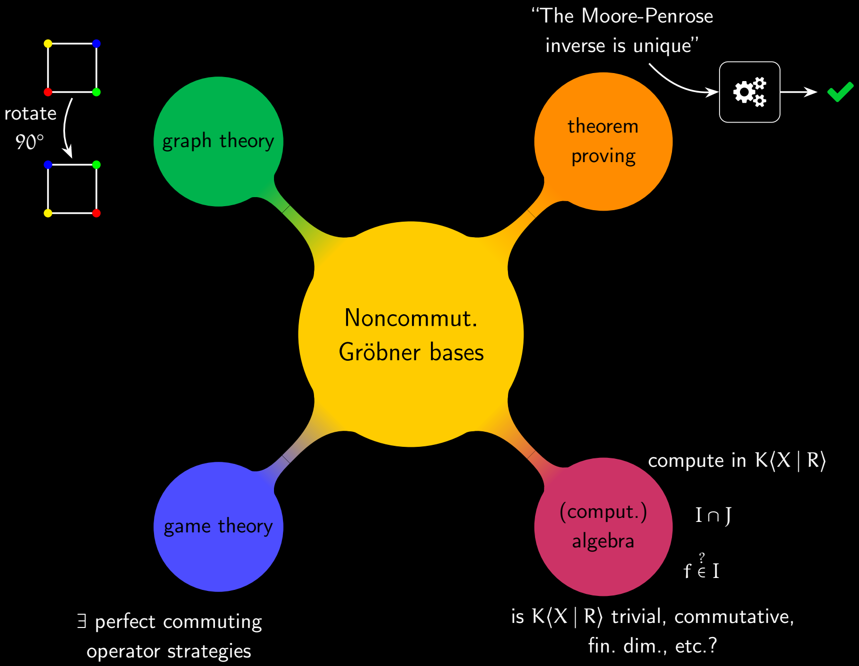

"The Moore-Penrose inverse is unique"
rotate 90°
graph theory
theorem proving
Noncommut. Gröbner bases
game theory
(comput.) algebra
compute in $K\langle X \mid R\rangle$
$I \cap J$
$f \overset{?}{\in} I$
$\exists$ perfect commuting operator strategies
is $K\langle X \mid R\rangle$ trivial, commutative, fin. dim., etc.?


Noncommut. Gröbner bases software
operator_gb (SageMath)
NCAlgebra (Mathematica)
Macaulay2
Opal (C++)
GBNP (GAP)
Letterplace (Singular)
bergman (Standard Lisp)
Oscar (Julia)
NCPoly (ApCoCoA)
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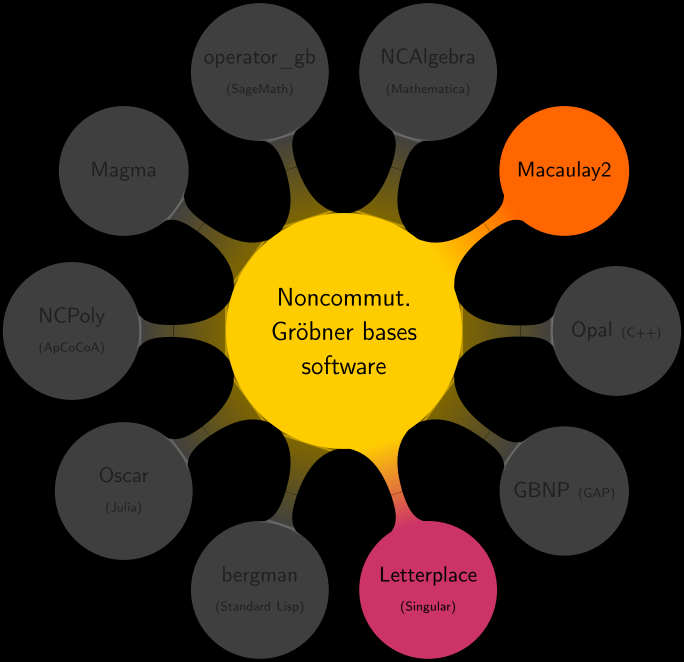
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An Example

Consider the ideal generated by $f_1, f_2 \in \mathbb{Q}\langle x, y, z\rangle$

$$f_1 = xy + yz \qquad\qquad f_2 = x^2 + xy - yx - y^2.$$

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```
> ideal II = x*y+y*z,x*x+x*y-y*x-y*y;
ideal GG = twostd(II);
2ss3s(2)s4(5)s(9)s(13)s(19)s(23)-5-s(25)s(34)s(38)s(45)s(50)s(61)-------6-s(64)s(78)s(85)s(95)s(108)-s(112)---s(127)s(133)--------
-----7-s(135)s(155)s(165)s(179)s(193)-s(198)---s(215)-s(220)------s(242)-s(248)--------------------8-s(249)s(276)s(290)s(309)s(334)
-s(344)---s(364)-s(370)-----s(394)-s(400)---------s(429)-s(436)------------------------9-s(440)s(475)s(494)s(519)s(550)-s(564)---s
(586)-s(594)-----s(621)-s(628)---------s(659)-s(666)-------------s(704)-s(712)------------(700)----------------------10-s(714)s(75
8)s(783)s(815)s(853)-s(872)---s(909)-s(924)-----s(955)-s(964)-------s(1000)-s(1008)--------(1000)-----s(1048)-s(1056)-------------
------s(1103)-s(1112)------------(1100)------------------------------11-s(1115)s(1169)s(1201)s(1241)s(1287)-s(1312)---s(1356)-s(13
76)-----s(1410)-s(1422)-------s(1462)-s(1472)-----------s(1517)-s(1526)-----------------s(1577)-s(1586)-------------------------s(
1644)-s(1654)-----------------------------------------------------12-s(1657)s(1722)s(1762)s(1811)s(1866)-s(1898)---s(1950)-s(1976)-
----s(2029)-s(2050)-------s(2095)-s(2108)--------(2100)-s(2159)-s(2170)---------------s(2226)-s(2236)---------------------s(2300)-
s(2310)----------(2300)---------------------s(2381)-s(2392)-----------------------------------------------------------------13-s(23
94)s(2471)s(2520)s(2579)s(2644)-s(2684)---s(2745)-s(2778)-----s(2839)-s(2866)-------s(2915)-s(2932)---------s(2988)-s(3002)--(3000
)-----------s(3064)-s(3076)---------------------s(3143)-s(3154)---------------------------s(3231)-s(3242)-------------------------
----------------s(3324)-s(3336)------------------------------------(3300)----------------------------------------14-s(3338)s(3428)
s(3487)s(3557)s(3633)-s(3682)---s(3753)-s(3794)-----s(3864)-s(3898)-------s(3970)-s(3998)---------s(4060)-s(4078)---------s(4147
)-s(4162)-----------------s(4237)-s(4250)-------------------------s(4332)-s(4344)-----------------------------------s(4434)-s(4446
)----------------------------------------------(4400)---s(4543)-s(4556)--------------------------------------------------------(45
00)--------------------------------15-s(4559)s(4663)s(4733)s(4815)s(4903)-s(4962)---s(5044)-s(5094)-----s(5174)-s(5216)-------s(52
97)-s(5332)---------s(5399)-s(5422)-----------s(5497)-s(5516)---------------s(5598)-s(5614)--------------(5600)---------s(5702)-s(
5716)----------------(5700)---------------s(5813)-s(5826)--------------------------(5800)-----------------s(5931)-s(5944)---------
-----------------------------------(5900)---------------s(6056)-s(6070)-----------------------------------------------------------
-----------(6000)----------------------------------16-s(6071)s(6190)s(6272)s(6367)s(6468)-s(6538)---s(6632)-s(6692)-----s(6783)-s(
6834)-------s(6925)-s(6968)---------s(7062)-s(7098)-----------s(7180)-s(7204)----(7200)---------s(7294)-s(7314)--------------(7300
)-----s(7411)-s(7428)----------------------------(7400)-s(7531)-s(7546)-------------------------------------s(7660)-s(7674)-------
--------------------------------------------s(7796)-s(7810)----------(7800)-------------------------------------------------------
------s(7937)-s(7952)----------------------------------------------------(7900)---------------------------------------------------
---------------17-s(7954)s(8089)s(8184)s(8293)s(8408)-s(8490)---s(8597)-s(8668)-----s(8771)-s(8832)-------s(8934)-s(8986)---------
s(9090)-s(9134)-----------s(9222)-s(9252)-------------s(9349)-s(9374)-----------------s(9479)-s(9500)-------------------------s(96
12)-s(9630)------------------------------(9600)-----s(9750)-s(9766)---------------------------------------------s(9897)-s(9912)---
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-----------------------------s(58516)-s(58540)----------------------------------------(58500)----------------------------------
--------------------------------------------------------------------(58400)-----------------------------------------------------
-------s(58874)-s(58898)----------------------------------------------------------------------------------------------------(588
00)------------------------------------------------------------------------------------------------------(58700)----------------
-----------------------s(59239)-s(59264)------------------------------------------------------------------(59200)---------------
--------------------------------------------------------------------------------------(59100)-----------------------------------
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-------27-s(59265)s(59615)s(59895)s(60199)s(60509)-s(60766)----s(61058)-s(61294)-----s(61572)-s(61788)--------s(62055)-s(62252)-
--------s(62511)-s(62690)-----------s(62944)-s(63106)-------(63100)-------s(63358)-s(63504)----(63500)------------s(63757)-s(638
88)-----------------s(64145)-s(64262)--------------------s(64526)-s(64630)---------------------s(64868)-s(64948)----------------
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---------------------(65900)----------------------s(66214)-s(66260)-------------------------------------------------------------
(66200)---s(66558)-s(66598)---------------------------------------------------------------------------------s(66909)-s(66944)---
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----------------------------------------------------------------------------------------------(67200)---------------------s(6763
6)-s(67664)-----------------------------------------------------------------(67600)---------------------------------------------
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---s(68389)-s(68414)---------------(68400)--------------------------------------------------------------------------------------
---------------(68300)------------------------------------------------------------------------------------------------------(682
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68700)------------------------------------------------------------------------------------------------------(68600)-------------
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```

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```

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-------------------------------------------------(351300)--------------------------------------------------------------------
-----------------------s(352009)-s(352044)---------------------------------------------(352000)------------------------------
---------------------------------------------------------------------(351900)------------------------------------------------
---------------------------------------------------(351800)------------------------------------------------------------------
---------------------------------(351700)---------------------------------------------------------------------------------s(
475)-s(352510)----------(352500)---------------------------------------------------------------------------------------------
------(352400)-----------------------------------------------------------------------------------------------------(352300)-
-------------------------------------------------------------------------------------------------(352200)--------------------
-------------------------------------------------------------------------------(352100)--------------------------------------
--------------------------------s(352941)-s(352976)--------------------------------------------------------------------------
(352900)-----------------------------------------------------------------------------------------------------(352800)--------
-------------------------------------------------------------------------------------------(352700)--------------------------
-------------------------------------------------------------------------(352600)--------------------------------------------
-------------------------------------------------------(352500)--------------------------------------------------------------
-s(353405)-s(353440)-----------------------------------------(353400)--------------------------------------------------------
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------------------------(353200)---------------------------------------------------------------------------------------------
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-------------------------------------------------------------------------------------------------(352900)--------------------
----------------------------------------------s(353863)-s(353898)------------------------------------------------------------
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)----------------------------------------------------------------------------------------------------(2200)--------------------
---------------------------------------------------------------------------------(2100)---------------------------------------
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---------------------(1800)---------------------------------------------------------------------------------------------------
-(1700)----------------------------------------------------------------------------------------------------(1600)-------------
---------------------------------------------------------------------------------------(1500)---------------------------------
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-----------------------------------------------(1300)-------------------------------------------------------------------------
---------------------------(1200)---------------------------------------------------------------------------------------------
-------(1100)----------------------------------------------------------------------------------------------------(1000)-------
---------------------------------------------------------------------------------------------(900)----------------------------
------------------------------------------------------------------------(800)-------------------------------------------------
---------------------------------------------------(700)----------------------------------------------------------------------
------------------------------(600)-------------------------------------------------------------------------------------------
---------(500)----------------------------------------------------------------------------------------------------(400)-------
---------------------------------------------------------------------------------------------(300)----------------------------
------------------------------------------------------------------------(200)-------------------------------------------------
---------------------------------------------------(100)----------------------------------------------------------------------
------------------------------
product criterion:0 chain criterion:1726044
shift V criterion:648631356
Auf Wiedersehen.
singular demo.sing  4511.08s user 21.85s system 64% cpu 1:56:58.99 total
(base) clemenshofstadler@Clemenss-MacBook-Air demo %
```

An Example

```
(base) clemenshofstadler@Clemenss-MacBook-Air demo % ./f4ncgb -m 10000 -d 75 demo.ms > demo.gb
[f4ncgb] ==== Input Parameters ====
[f4ncgb] Characteristic:    0
[f4ncgb] Max. Iterations:   10000
[f4ncgb] Max. amb. degree:  75
[f4ncgb] Monomial order:    z < y < x
[f4ncgb] Nr. threads:       1
[f4ncgb] Output file:       None. Writing output to console.
[f4ncgb] Proof logging:     off
[f4ncgb] Tracer:            on
[f4ncgb] PID of F4NCGB:     11942
[f4ncgb] ==== Starting Gröbner Basis Computation ====
```

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[f4ncgb] PID of F4NCGB:     11942
[f4ncgb] ==== Starting Gröbner Basis Computation ====
[f4ncgb] ==== Basis computation finished ====
[f4ncgb]
[f4ncgb] maximum resident set size:              245744.00 MB
[f4ncgb] store find calls:                         13478136
[f4ncgb] store find hits:                           88.36 %
[f4ncgb] parse input:                                0.00 (0.00 %)
[f4ncgb] computing ambiguities:                      2.06 (25.76 %)
[f4ncgb]   computing overlaps:                       0.34 (4.20 %)
[f4ncgb]   computing inclusions:                     0.04 (0.52 %)
[f4ncgb] handling critical pairs:                    0.44 (5.47 %)
[f4ncgb] symbolic preprocessing:                     0.42 (5.26 %)
[f4ncgb] linear algebra:                             4.28 (53.44 %)
[f4ncgb]   Gauss elimination:                        3.95 (49.37 %)
[f4ncgb]     reduce (CPU-time):                      3.81 (47.60 %) /#t: 3.81 (47.60 %)
[f4ncgb]   CRT:                                      0.00 (0.02 %)
[f4ncgb]   rat. reconstruction:                      0.00 (0.02 %)
[f4ncgb] construct new elements:                     0.02 (0.24 %)
[f4ncgb] other:                                      0.00 (0.00 %)
[f4ncgb]
[f4ncgb] total process time:                         8.01 seconds
(base) clemenshofstadler@Clemenss-MacBook-Air demo %
```

Example	Letterplace	f4ncgb 1 core	f4ncgb 4 cores	f4ncgb 16 cores
`4nilp5s-10`	1282	150	79	63
`braid3-16`	18 953	105	34	18
`braidXY-12`	1847	62	52	52
`holt_G3562h-17`	>43 200	25 021	12 671	6824
`lascala_neuh-13`	171	9	5	4
`lp1-15`	24 166	266	179	155
`lv2d10-100`	>43 200	48	27	47
`malle_G12h-100`	4142	89	74	73

(Timings in sec)


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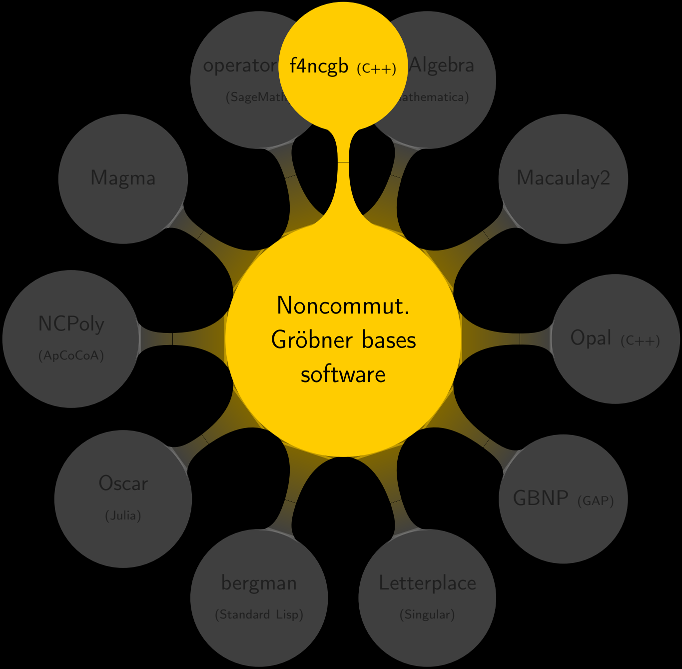
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f4ncgb

Open-source C++ library that ports commutative advancements to the noncommutative setting.

- Gröbner basis computation in $\mathbb{Q}\langle X\rangle$ and $\mathbb{Z}_p\langle X\rangle$ for prime $p < 2^{31}$
- Several orders of magnitude faster than current state of the art
- Proof logging via cofactor representations
- Also part of OSCAR SYMBOLIC TOOLS

Data structures

- Monomials are shared
- Coefficients are shared
- Prefix tree for divisions

Algorithms

- Noncomm. F4 algorithm
- Sparse linear algebra (multi-modular, parallelized, probabilistic)
- Proof logging

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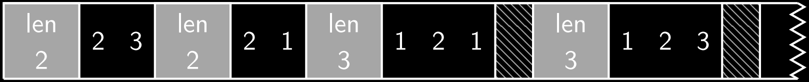
f4ncgb

Monomials & Polynomials

Represent vars by index according to mon. order: $x_1 < x_2 < \cdots < x_n$

$$\begin{array}{ccccccc} x_1 & < & x_2 & < & \cdots & < & x_n \\ \downarrow & & \downarrow & & & & \downarrow \\ 1 & & 2 & & & & n \end{array}$$



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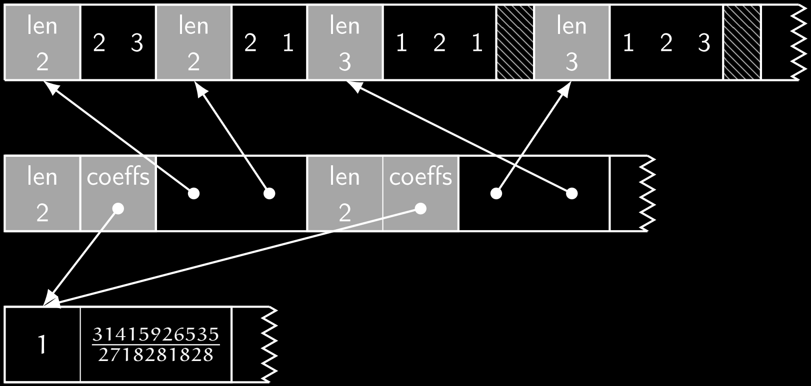

Monomial Divisibility Tests

Observation: divisor candidates are always the same (lm's of the GB)

Exploit this information $\rightsquigarrow$ keep prefix tree of all leading monomials

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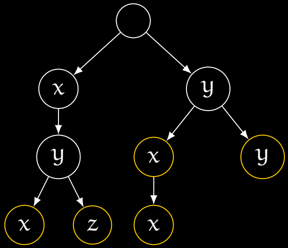
$$G = \{\, xyx - \ldots,\; xyz + \ldots,\; yx + \ldots,\; yxx - \ldots,\; yy + \ldots \}$$



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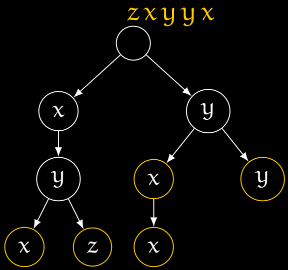

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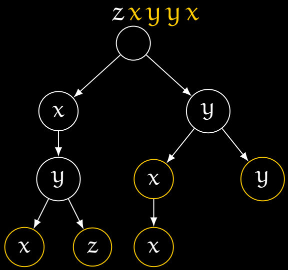

Monomial Divisibility Tests

Observation: divisor candidates are always the same (lm's of the GB)

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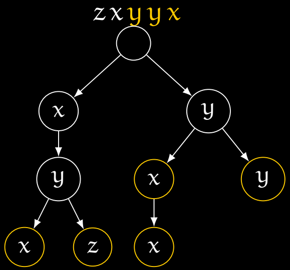

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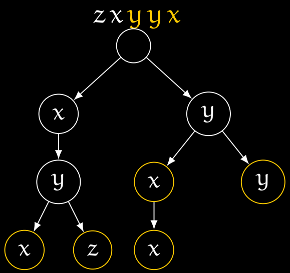

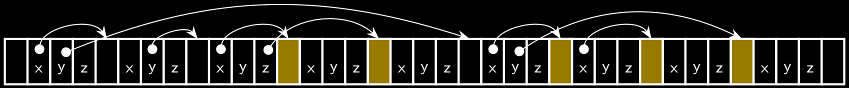

Data structures

- Monomials are shared
- Coefficients are shared
- Prefix tree for divisions

Algorithms

- Noncomm. F4 algorithm
- Sparse linear algebra (multi-modular, parallelized, probabilistic)
- Proof logging

f4ncgb

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Given input $f_1, \dots, f_r$, write each $g \in GB$ as

$$g = \sum_j p_j \cdot f_j \cdot q_j$$

"*cofactor representation*"

with $p_j, q_j \in K\langle X \rangle$.

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Cofactor representations certify ideal membership.

Can be computed during Gaussian elimination:

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$$T \cdot \begin{pmatrix} \text{—} & p_1 & \text{—} \\ & \vdots & \\ \text{—} & p_k & \text{—} \end{pmatrix} = \text{RRef}$$

The rows of T give cofactor representations of g_i in terms of $f_1, \ldots, f_r$ and $g_1, \ldots, g_{i-1}$.

Substitution yields representations w.r.t. $f_1, \ldots, f_r$ (⚠ exp. blowup!)

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Open-source C++ library that ports commutative advancements to the noncommutative setting.

- Gröbner basis computation in $\mathbb{Q}\langle X\rangle$ and $\mathbb{Z}_p\langle X\rangle$ for prime $p < 2^{31}$
- Several orders of magnitude faster than current state of the art
- Proof logging via cofactor representations
- Also part of OSCAR SYMBOLIC TOOLS