# Algebraic First-Order Theorem Proving

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based on joint work with P. Krug, C.G. Raab, G. Regensburger, and T. Verron

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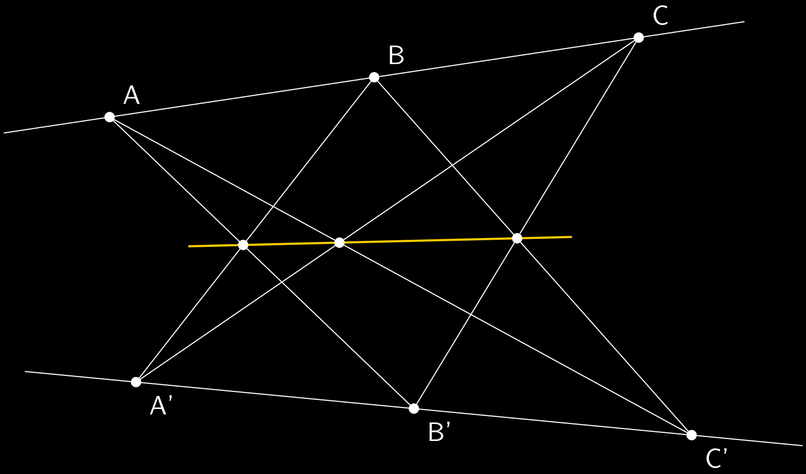
Der Wissenschaftsfonds.


A
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DISCRETE MATHEMATICS AND ITS APPLICATIONS
Series Editor KENNETH H. ROSEN

# HANDBOOK OF LINEAR ALGEBRA

## SECOND EDITION

$$\begin{bmatrix} 2 & 2 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

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**Leslie Hogben**

CRC Press
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A CHAPMAN & HALL BOOK

## 5.7 Pseudo-Inverse

**Definitions:**

A **Moore–Penrose pseudo-inverse** of a matrix $A \in \mathbb{C}^{m\times n}$ is a matrix $A^\dagger \in \mathbb{C}^{n\times m}$ that satisfies the following four **Penrose** conditions:

$$AA^\dagger A = A; \quad A^\dagger AA^\dagger = A^\dagger; \quad (AA^\dagger)^* = AA^\dagger; \quad (A^\dagger A)^* = A^\dagger A.$$

**Facts:**

All the following facts except those with a specific reference can be found in [Gra83, pp. 105–141] or [RM71, pp. 44–67].

1. Every $A \in \mathbb{C}^{m\times n}$ has a unique pseudo-inverse $A^\dagger$.
2. If $A \in \mathbb{R}^{m\times n}$, then $A^\dagger$ is real.
3. If $A \in \mathbb{C}^{m\times n}$ of rank $r$ has a full rank decomposition $A = BC$, where $B \in \mathbb{C}^{m\times r}$ and $C \in \mathbb{C}^{r\times n}$, then $A^\dagger$ can be evaluated using $A^\dagger = C^*(B^*AC^*)^{-1}B^*$.
4. [LH95, p. 38] If $A \in \mathbb{C}^{m\times n}$ of rank $r \le \min\{m,n\}$ has an SVD $A = U\Sigma V^*$, then its pseudo-inverse is $A^\dagger = V\Sigma^\dagger U^*$, where
$$\Sigma^\dagger = \operatorname{diag}(1/\sigma_1, \ldots, 1/\sigma_r, 0, \ldots, 0) \in \mathbb{R}^{n\times m}.$$
5. [Hig96, p. 412] The pseudo-inverse $A^\dagger$ of $A \in F^{m\times n}$ ($F = \mathbb{C}$ or $\mathbb{R}$) solves the minimization problem
$$\min_{X\in F^{n\times m}} \|AX - I_m\|_F^2.$$
6. $\mathbf{0}_{mn}^\dagger = \mathbf{0}_{nm}$ and $J_{mn}^\dagger = \frac{1}{mn}J_{nm}$, where $\mathbf{0}_{mn} \in \mathbb{C}^{m\times n}$ is the all 0s matrix and $J_{mn} \in \mathbb{C}^{m\times n}$ is the all 1s matrix.
7. If $\mathbf{x} \neq \mathbf{0}$, $\mathbf{y} \neq \mathbf{0}$, then $(\mathbf{x}\mathbf{y}^*)^\dagger = \dfrac{\mathbf{y}\mathbf{x}^*}{\|\mathbf{x}\|^2\|\mathbf{y}\|^2}$.
8. If $\mathbf{x} \neq \mathbf{0}$, then $\mathbf{x}^\dagger = \dfrac{\mathbf{x}^*}{\|\mathbf{x}\|^2}$.
9. Let $\alpha$ be a scalar. Denote
$$\alpha^\dagger = \begin{cases} \alpha^{-1}, & \text{if } \alpha \neq 0, \\ 0, & \text{if } \alpha = 0. \end{cases}$$
Then
   (a) $(\alpha A)^\dagger = \alpha^\dagger A^\dagger$.
   (b) $(\operatorname{diag}(\beta_1, \beta_2, \cdots, \beta_n))^\dagger = \operatorname{diag}(\beta_1^\dagger, \beta_2^\dagger, \cdots, \beta_n^\dagger)$.
10. $(A^\dagger)^* = (A^*)^\dagger$; $(A^\dagger)^\dagger = A$.
11. If $A$ is a nonsingular square matrix, then $A^\dagger = A^{-1}$.
12. If $U$ has orthonormal columns or orthonormal rows, then $U^\dagger = U^*$.
13. If $A = A^*$ and $A = A^2$, then $A^\dagger = A$.
14. $A^\dagger = A^*$ if and only if $A^*A$ is idempotent.
15. If $A$ is normal and $k$ is a positive integer, then $AA^\dagger = A^\dagger A$ and $(A^k)^\dagger = (A^\dagger)^k$.
16. If $U \in \mathbb{C}^{m\times n}$ is of rank $n$ and satisfies $U^\dagger = U^*$, then $U$ has orthonormal columns.
17. If $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary matrices, then $(UAV)^\dagger = V^*A^\dagger U^*$.
18. $A^\dagger = (A^*A)^\dagger A^* = A^*(AA^*)^\dagger$. In particular,
   (a) if $A \in \mathbb{C}^{m\times n}$ $(m \ge n)$ has full rank $n$, then $A^\dagger = (A^*A)^{-1}A^*$;
   (b) if $A \in \mathbb{C}^{m\times n}$ $(m \le n)$ has full rank $m$, then $A^\dagger = A^*(AA^*)^{-1}$.
19. Let $A \in \mathbb{C}^{m\times n}$. Then
   (a) $A^\dagger A$, $AA^\dagger$, $I_n - A^\dagger A$, and $I_m - AA^\dagger$ are orthogonal projections.
   (b) $\operatorname{rank}(A) = \operatorname{rank}(A^\dagger) = \operatorname{rank}(AA^\dagger) = \operatorname{rank}(A^\dagger A)$.
   (c) $\operatorname{rank}(I_n - A^\dagger A) = n - \operatorname{rank}(A)$.
   (d) $\operatorname{rank}(I_m - AA^\dagger) = m - \operatorname{rank}(A)$.
20. $AA^\dagger = \operatorname{Proj}_{\operatorname{range}(A)}$; $A^\dagger A = \operatorname{Proj}_{\operatorname{range}(A^\dagger)}$.
21. Suppose that $A \in F^{m\times n}$, where $F = \mathbb{C}$ or $\mathbb{R}$. Then
   (a) $\operatorname{range}(A) = \operatorname{range}(AA^*) = \operatorname{range}(AA^\dagger)$.
   (b) $\operatorname{range}(A^\dagger) = \operatorname{range}(A^*) = \operatorname{range}(A^*A) = \operatorname{range}(A^\dagger A)$.
   (c) $\ker(A) = \ker(A^*A) = \ker(A^\dagger A)$.
   (d) $\ker(A^\dagger) = \ker(A^*) = \ker(AA^*) = \ker(AA^\dagger)$.
   (e) $\operatorname{range}(A^\dagger A) \oplus \ker(A^\dagger A) = F^n$.
   (f) $\operatorname{range}(AA^\dagger) \oplus \ker(AA^\dagger) = F^m$.
22. If $A = A_1 + A_2 + \cdots + A_k$, $A_i^*A_j = 0$, and $A_iA_j^* = 0$, for all $i, j = 1, \cdots, k$, $i \neq j$, then $A^\dagger = A_1^\dagger + A_2^\dagger + \cdots + A_k^\dagger$.
23. If $A$ is an $m\times r$ matrix of rank $r$ and $B$ is an $r\times n$ matrix of rank $r$, then $(AB)^\dagger = B^\dagger A^\dagger$.
24. $(A^*A)^\dagger = A^\dagger(A^*)^\dagger$; $(AA^*)^\dagger = (A^*)^\dagger A^\dagger$.
25. [Gre66] Each one of the following conditions is necessary and sufficient for $(AB)^\dagger = B^\dagger A^\dagger$:
   (a) $\operatorname{range}(BB^*A^*) \subseteq \operatorname{range}(A^*)$ and $\operatorname{range}(A^*AB) \subseteq \operatorname{range}(B)$.
   (b) $A^\dagger ABB^*$ and $A^*ABB^\dagger$ are both Hermitian matrices.
   (c) $A^\dagger ABB^*A^* = BB^*A^*$ and $BB^\dagger A^*AB = A^*AB$.
   (d) $A^\dagger ABB^*A^*ABB^\dagger = BB^*A^*A$.
   (e) $A^\dagger AB = B(AB)^\dagger AB$ and $BB^\dagger A^* = A^*AB(AB)^\dagger$.
26. $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$, where $\otimes$ denotes the Kronecker product.
27. $A^\dagger = \lim_{\alpha\to 0} A^*(\alpha I + AA^*)^{-1} = \lim_{\alpha\to 0}(\alpha I + A^*A)^{-1}A^*$.
28. $A^\dagger = \sum_{j=1}^{\infty} A^*(I + AA^*)^{-j} = \sum_{j=1}^{\infty}(I + A^*A)^{-j}A^*$.
29. (Continuity of pseudo-inverse) Suppose that $A \in F^{m\times n}$ and $E \in F^{m\times n}$, where $F = \mathbb{C}$ or $\mathbb{R}$. Then $\lim_{E\to 0}(A+E)^\dagger = A^\dagger$ if and only if there is $\epsilon > 0$ such that $\operatorname{rank}(A+E) = \operatorname{rank}(A)$ when $\|E\|_2 \le \epsilon$.
30. Let $A \in \mathbb{C}^{m\times n}$ be of rank $r$ where $0 < r < \min\{m,n\}$. Suppose that $A$ can be partitioned as
$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$
where $A_{11} \in \mathbb{C}^{r\times r}$ and $\operatorname{rank}(A_{11}) = r$. Then
$$A^\dagger = \begin{bmatrix} A_{11}^*XA_{11}^* & A_{11}^*XA_{21}^* \\ A_{12}^*XA_{11}^* & A_{12}^*XA_{21}^* \end{bmatrix},$$
where
$$X = (A_{11}A_{11}^* + A_{12}A_{12}^*)^{-1}A_{11}(A_{11}^*A_{11} + A_{21}^*A_{21})^{-1}.$$

# Reverse order law for the Moore–Penrose inverse ☆

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ABSTRACT

In this paper we present new results related to the reverse order law for the Moore–Penrose inverse of operators on Hilbert spaces. Some finite-dimensional results are extended to infinite-dimensional settings.



## 1. Introduction

In this paper we extend some results from [15] to infinite-dimensional settings. Among other things, we obtain the reverse order law for the Moore–Penrose inverse as a corollary. We use the matrix form of a linear bounded operator, and this matrix form is induced by some natural decompositions of Hilbert spaces.

In the rest of the Introduction we formulate two auxiliary results. In Section 2 we present the results related to the reverse order rule for the Moore–Penrose inverse of Hilbert space operators with closed range. The present paper is the extension of results from [15] to infinite-dimensional settings.

## 2. Reverse order law

In this section we prove the results concerning the reverse order law for the Moore–Penrose inverse.

**Theorem 2.2.** *Let $X$, $Y$, $Z$ be Hilbert spaces, and let $A \in \mathcal{L}(Y,Z)$, $B \in \mathcal{L}(X,Y)$ be such that $A$, $B$, $AB$ have closed ranges. Then the following statements hold:*

(a) $AB(AB)^\dagger = ABB^\dagger A^\dagger \Leftrightarrow A^*AB = BB^\dagger A^*AB \Leftrightarrow \mathcal{R}(A^*AB) \subseteq \mathcal{R}(B) \Leftrightarrow B^\dagger A^\dagger \in (AB)\{1,2,3\}$;
(b) $(AB)^\dagger AB = B^\dagger A^\dagger AB \Leftrightarrow ABB^* = ABB^*A^\dagger A \Leftrightarrow \mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*) \Leftrightarrow B^\dagger A^\dagger \in (AB)\{1,2,4\}$;
(c) *The following statements are equivalent:*
  (1) $(AB)^\dagger = B^\dagger A^\dagger$;
  (2) $AB(AB)^\dagger = ABB^\dagger A^\dagger$ *and* $(AB)^\dagger AB = B^\dagger A^\dagger AB$;
  (3) $A^*AB = BB^\dagger A^*AB$ *and* $ABB^* = ABB^*A^\dagger A$;
  (4) $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ *and* $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$.

**Proof.** The operators $A$ and $B$ have the same matrix representations as in the previous theorem. The following products will be useful:

$$AB = \begin{bmatrix} A_1B_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad (AB)^\dagger = \begin{bmatrix} (A_1B_1)^\dagger & 0 \\ 0 & 0 \end{bmatrix}, \qquad B^\dagger A^\dagger = \begin{bmatrix} B_1^{-1}A_1^*D^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

First, we find the equivalent expressions for our statements in terms of $A_1$, $A_2$ and $B_1$.



(a) 1. $AB(AB)^\dagger = ABB^\dagger A^\dagger \Leftrightarrow A_1B_1(A_1B_1)^\dagger = A_1A_1^*D^{\ 1}$. Here $A_1B_1(A_1B_1)^\dagger$ is Hermitian, so $[A_1A_1^*, D^{\ 1}] = 0$.
  2. $A^*AB = BB^\dagger A^*AB \Leftrightarrow A_2^*A_1 = 0$.
  3. Notice that $\mathcal{R}(A^*AB) \subset \mathcal{R}(B)$ if and only if $BB^\dagger A^*AB = A^*AB$, so $2 \Leftrightarrow 3$.
  4. If we check properly the Penrose equations, then we see that: $B^\dagger A^\dagger \in (AB)\{1,2,3\} \Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1$ and $[A_1A_1^*, D^{-1}] = 0$.

Now, we prove the following: $1 \Leftrightarrow 2$, $4 \Rightarrow 2$ and $1 \Rightarrow 4$.
We prove $1 \Leftrightarrow 2$. Notice that

$$A_1B_1(A_1B_1)^\dagger = A_1A_1^*D^{-1} \quad \Leftrightarrow \quad (A_1B_1)^\dagger = (A_1B_1)^\dagger A_1A_1^*D^{-1}.$$

The last statement is obtained by multiplying the first expression by $(A_1B_1)^\dagger$ from the left side, or multiplying the second expression by $A_1B_1$ from the left side, and using $A_1A_1^* = A_1B_1B_1^{-1}A_1^*$. Now, there is a chain of the equivalences:

$$\begin{aligned} (A_1B_1)^\dagger = (A_1B_1)^\dagger A_1A_1^*D^{-1} \quad &\Leftrightarrow \quad (A_1B_1)^\dagger\big(A_1A_1^* + A_2A_2^*\big) = (A_1B_1)^\dagger A_1A_1^* \\ &\Leftrightarrow \quad (A_1B_1)^\dagger A_2A_2^* = 0 \quad \Leftrightarrow \quad \mathcal{R}\big(A_2A_2^*\big) \subset \mathcal{N}\big((A_1B_1)^\dagger\big) \\ &\Leftrightarrow \quad \mathcal{R}(A_2) \subset \mathcal{N}\big((A_1B_1)^*\big) \quad \Leftrightarrow \quad B_1^*A_1^*A_2 = 0 \quad \Leftrightarrow \quad A_1^*A_2 = 0. \end{aligned}$$

Therefore, we have just proved that $1 \Leftrightarrow 2$.

Now we prove $1 \Rightarrow 4$. If we multiply $A_1B_1(A_1B_1)^\dagger = A_1A_1^*D^{-1}$ by $A_1B_1$ from the right side, we get $A_1A_1^*D^{-1}A_1 = A_1$. Thus, 4 holds.

Finally, we prove $4 \Rightarrow 2$. If $A_1A_1^*D^{-1}A_1 = A_1$ and $[A_1A_1^*, D^{-1}] = 0$, then $A_1A_1^*A_1 = DA_1 = A_1A_1^*A_1 + A_2A_2^*A_1$, implying that $A_2A_2^*A_1 = 0$. Hence, $\mathcal{R}(A_1) \subset \mathcal{N}(A_2A_2^*) = \mathcal{N}(A_2^*)$, so $A_2^*A_1 = 0$. Thus, 2 holds.

Notice that the equivalence $3 \Leftrightarrow 4$ is proved in [8], also.

(b) 1. $(AB)^\dagger AB = B^\dagger A^\dagger AB \Leftrightarrow (A_1B_1)^\dagger A_1B_1 = B_1^{-1}A_1^*D^{-1}A_1B_1$. Moreover, $(A_1B_1)^\dagger A_1B_1$ is Hermitian, so $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$.
  2. $ABB^* = ABB^*A^\dagger A \Leftrightarrow A_1B_1B_1^*A_1^*D^{-1}A_1 = A_1B_1B_1^*$ and $A_1B_1B_1^*A_1^*D^{-1}A_2 = 0$.
  3. Notice that $\mathcal{R}(BB^*A^*) \subset \mathcal{R}(A^*)$ if and only if $A^\dagger ABB^*A^* = BB^*A^*$, which is equivalent to $ABB^*A^\dagger A = ABB^*$. Hence, $2 \Leftrightarrow 3$.
  4. The Penrose equations imply that: $B^\dagger A^\dagger \in (AB)\{1,2,4\} \Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1$ and $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$.

We prove $1 \Rightarrow 4 \Rightarrow 2 \Rightarrow 1$.

Suppose that 1 holds. If we multiply $(A_1B_1)^\dagger A_1B_1 = B_1^{-1}A_1^*D^{-1}A_1B_1$ by $A_1B_1$ from the left side, we obtain $A_1 = A_1A_1^*D^{-1}A_1$. Furthermore, $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$ holds. Therefore, $1 \Rightarrow 4$.

Suppose that 4 holds. Obviously, $A_1B_1B_1^*A_1^*D^{-1}A_1 = A_1A_1^*D^{-1}A_1B_1B_1^* = A_1B_1B_1^*$. Thus, the first equality of 2 holds. The second equality of 2 also holds, since $A_1^*D^{-1}A_2 = 0 \Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1$, which is shown in the proof of Theorem 2.1. Here we use again $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$. Consequently, $4 \Rightarrow 2$.

In order to prove that $2 \Rightarrow 1$, we multiply $A_1B_1B_1^*A_1^*D^{-1}A_1 = A_1B_1B_1^*$ by $(A_1B_1)^\dagger$ from the left side. It follows that $B_1^*A_1^*D^{-1}A_1 = (A_1B_1)^\dagger A_1B_1B_1^*$, so $(A_1B_1)^\dagger A_1B_1 = B_1^*A_1^*D^{-1}A_1(B_1^*)^{-1}$ which is equivalent to $(A_1B_1)^\dagger A_1B_1 = B_1^{-1}A_1^*D^{-1}A_1B_1$. Hence, $2 \Rightarrow 1$.

Notice that $3 \Leftrightarrow 4$ is also proved in [8].

Finally, the part (c) follows from the parts (a) and (b). □

We also prove the following result.

**Theorem 2.3.** *Let $X$, $Y$, $Z$ be Hilbert spaces, and let $A \in \mathcal{L}(Y,Z)$, $B \in \mathcal{L}(X,Y)$ be such that $A$, $B$, $AB$ have closed ranges. Then we have:*

(a) $AB(AB)^\dagger A = ABB^\dagger \Leftrightarrow A^*ABB^\dagger = BB^\dagger A^*A \Leftrightarrow \mathcal{R}(A^*AB) \subseteq \mathcal{R}(B) \Leftrightarrow B^\dagger A^\dagger \in (AB)\{1,2,3\}$;
(b) $B(AB)^\dagger AB = A^\dagger AB \Leftrightarrow A^\dagger ABB^* = BB^*A^\dagger A \Leftrightarrow \mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*) \Leftrightarrow B^\dagger A^\dagger \in (AB)\{1,2,4\}$;
(c) *The following three statements are equivalent:*
  (1) $(AB)^\dagger = B^\dagger A^\dagger$;
  (2) $AB(AB)^\dagger A = ABB^\dagger$ *and* $B(AB)^\dagger AB = A^\dagger AB$;
  (3) $A^*ABB^\dagger = BB^\dagger A^*A$ *and* $A^\dagger ABB^* = BB^*A^\dagger A$.

**Proof.** The operators $A$ and $B$ have the same matrix representations as in the previous theorem. First, we find equivalent expressions, in the terms of $A_1$, $A_2$ and $B_1$, for our assumptions.

$$
\begin{tikzcd}
 & \ker(a) \arrow[r] \arrow[d] & \ker(b) \arrow[r] \arrow[d] & \ker(c) \arrow[d] \arrow[dddll, rounded corners, to path={-- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south |- \tikztostart.south) -| (\tikztotarget)}] & \\
 & A \arrow[r, "f"'] \arrow[d, "a"] & B \arrow[r, "g"'] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"'] \arrow[d] & B' \arrow[r, "g'"'] \arrow[d] & C' \arrow[d] & \\
 & \operatorname{coker}(a) \arrow[r] & \operatorname{coker}(b) \arrow[r] & \operatorname{coker}(c) &
\end{tikzcd}
$$

*"The Moore-Penrose inverse is unique"*

Def.: A matrix $B$ is Moore-Penrose inverse of a matrix $A$ if

$$ABA = A, \qquad BAB = B, \qquad B^*A^* = AB, \qquad A^*B^* = BA$$

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Claim If $B$ and $C$ satisfy these identities, then $B = C$.

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**Proof**

$$B = BAB = BACAB = \ldots = C$$

**An algebraic point of view**

$$L = R \iff L - R = 0$$

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# Noncommutative polynomials

Noncom. polynomial $f = c_1 \cdot w_1 + \cdots + c_d \cdot w_d$

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words over $X = \{x_1, \ldots, x_n\}$

# Noncommutative polynomials

Noncom. polynomial $$f = \overbrace{\boxed{c_1}}^{\text{Integers}} \cdot \underbrace{\boxed{w_1}}_{\text{words over } X = \{x_1, \ldots, x_n\}} + \cdots + \boxed{c_d} \cdot \boxed{w_d}$$

Example $2xy + yx - 2xzx + 4$

# Noncommutative polynomials

Integers $\qquad$ free algebra

Noncom. polynomial $\quad f = \boxed{c_1} \cdot \boxed{w_1} + \cdots + \boxed{c_d} \cdot \boxed{w_d} \quad \in \boxed{\mathbb{Z}\langle X \rangle}$

words over $X = \{x_1, \ldots, x_n\}$

Example $\quad 2xy + yx - 2xzx + 4 \ \in \ \mathbb{Z}\langle x, y, z \rangle$

# Noncommutative polynomials

Integers                                                    free algebra

Noncom. polynomial     $f = \boxed{c_1} \cdot \boxed{w_1} + \cdots + \boxed{c_d} \cdot \boxed{w_d} \quad \in \boxed{\mathbb{Z}\langle X \rangle}$

words over $X = \{x_1, \ldots, x_n\}$

Example     $2xy + yx - 2xzx + 4 \ \in \ \mathbb{Z}\langle x, y, z\rangle$

Arithmetic operations

$$\begin{aligned}
\text{Addition} \quad &= \quad \text{like in the commutative case} \\
(xy - z) + (yx + 2z) \quad &= \quad xy + yx + z \\[1em]
\text{Multiplication} \quad &= \quad \text{concatenation of words} \\
(xy - z) \cdot (yx + 2z) \quad &= \quad xyyx + 2xyz - zyx - 2zz
\end{aligned}$$

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$$ABA - A, \qquad BAB - B, \qquad B^*A^* - AB, \qquad A^*B^* - BA$$

**Claim** If $B$ and $C$ satisfy these identities, then $B = C$.

**Proof**

$$B = BAB = BACAB = \ldots = C$$

**An algebraic point of view**

$$L = R \iff L - R$$

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$$aba - a\,, \qquad bab - b, \qquad b^*a^* - ab\,, \qquad a^*b^* - ba$$

Claim If $B$ and $C$ satisfy these identities, then $B = C$.

Proof $$B = BAB = BACAB = \ldots = C$$

An algebraic point of view

$$L = R \quad \Longleftrightarrow \quad l - r \in \mathbb{Z}\langle \mathbf{X} \rangle$$

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Claim If $B$ and $C$ satisfy these identities, then $B = C$.

Proof $$B = BAB = BACAB = \ldots = C$$

An algebraic point of view

$$L = R \quad \Longleftrightarrow \quad l - r \in \mathbb{Z}\langle \mathbf{X} \rangle$$
$$B = \ldots = C \quad \Longleftrightarrow \quad ?$$

## Consequences

**Definition** A nonempty set $I \subseteq \mathbb{Z}\langle X \rangle$ is a (two-sided) ideal if

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- if $f \notin (f_1, \ldots, f_r)$, we might run into an infinite computation

## *"The Moore-Penrose inverse is unique"*

Def.: A matrix $B$ is Moore-Penrose inverse of a matrix $A$ if

$$aba - a, \qquad bab - b, \qquad b^*a^* - ab, \qquad a^*b^* - ba$$

Claim If $B$ and $C$ satisfy these identities, then $B = C$.

Proof $$B = BAB = BACAB = \ldots = C$$

An algebraic point of view

$$L = R \iff l - r \in \mathbb{Z}\langle \mathbf{X} \rangle$$

$$B = \ldots = C \iff ?$$

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Proof Using our software package operator_gb...

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sage: from operator_gb import *
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**Observation** Proof only relies on basic linearity properties

$\Rightarrow$ Statement proven for matrices, (un)bounded operators, morphisms,...

# Operator statements

**Operators**

- $0, A, B, C, \dots$
- $S + T, \ \ S \cdot T, \ \ f(T_1, \dots, T_n)$

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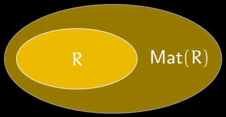

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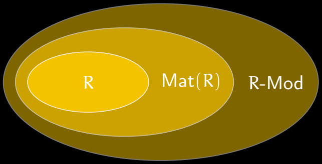

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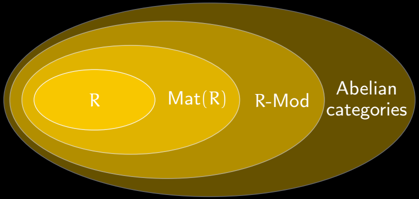

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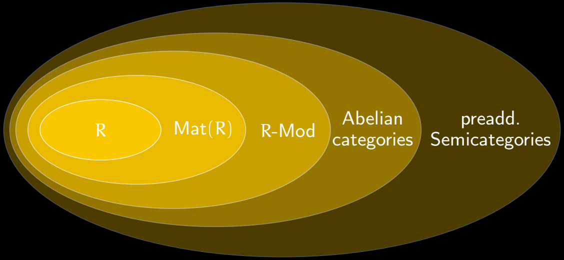

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$$S = T, \quad \neg\,\varphi, \quad (\varphi \wedge \psi), \quad (\varphi \vee \psi), \quad (\varphi \Rightarrow \psi), \quad \exists X : \varphi, \quad \forall X : \varphi$$

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Fact: Determining universal truth is not decidable

Best we can hope for: semi-decision procedure

# Semi decision procedure

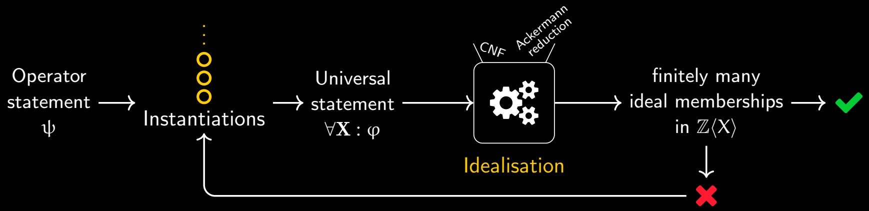
**Quasi-identities**

(Helton, Stankus, Wavrik '98, Schmitz, Levandovskyy '20, Raab, Regensburger, Hossein Poor '21)

$$\bigwedge_{i=1}^{m} P_i = Q_i \;\Rightarrow\; S = T \qquad \text{iff} \qquad s - t \in \left(p_1 - q_1, \ldots, p_m - q_m\right)$$

# Semi decision procedure

General operator statements



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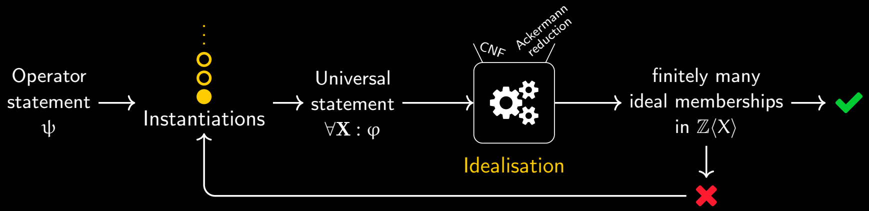

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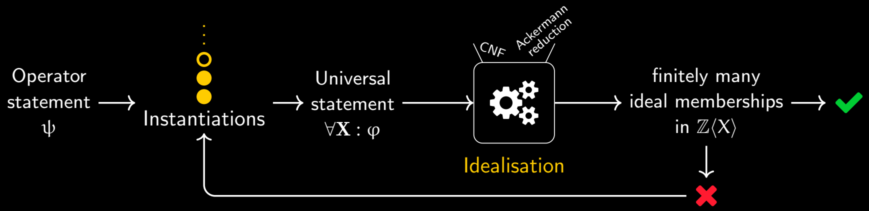

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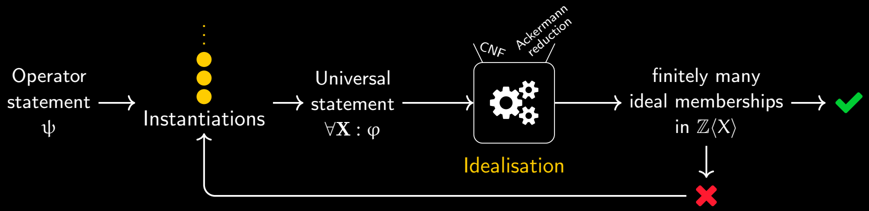

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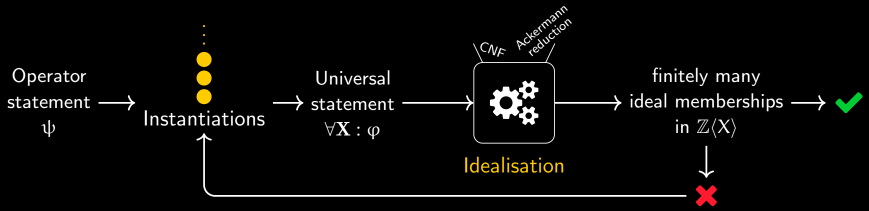


Theorem (H., Raab, Regensburger '22)

An operator statement is universally true iff this procedure terminates and returns ✔.

## 5.7 Pseudo-Inverse

**Definitions:**

A **Moore–Penrose pseudo-inverse** of a matrix $A \in \mathbb{C}^{m\times n}$ is a matrix $A^\dagger \in \mathbb{C}^{n\times m}$ that satisfies the following four **Penrose** conditions:

$$AA^\dagger A = A; \quad A^\dagger AA^\dagger = A^\dagger; \quad (AA^\dagger)^* = AA^\dagger; \quad (A^\dagger A)^* = A^\dagger A.$$

**Facts:**

All the following facts except those with a specific reference can be found in [Gra83, pp. 105–141] or [RM71, pp. 44–67].

1. Every $A \in \mathbb{C}^{m\times n}$ has a unique pseudo-inverse $A^\dagger$.
2. If $A \in \mathbb{R}^{m\times n}$, then $A^\dagger$ is real.
3. If $A \in \mathbb{C}^{m\times n}$ of rank $r$ has a full rank decomposition $A = BC$, where $B \in \mathbb{C}^{m\times r}$ and $C \in \mathbb{C}^{r\times n}$, then $A^\dagger$ can be evaluated using $A^\dagger = C^*(B^*AC^*)^{-1}B^*$.
4. [LH95, p. 38] If $A \in \mathbb{C}^{m\times n}$ of rank $r \le \min\{m,n\}$ has an SVD $A = U\Sigma V^*$, then its pseudo-inverse is $A^\dagger = V\Sigma^\dagger U^*$, where

$$\Sigma^\dagger = \operatorname{diag}(1/\sigma_1, \dots, 1/\sigma_r, 0, \dots, 0) \in \mathbb{R}^{n\times m}.$$

5. [Hig96, p. 412] The pseudo-inverse $A^\dagger$ of $A \in F^{m\times n}$ ($F = \mathbb{C}$ or $\mathbb{R}$) solves the minimization problem

$$\min_{X\in F^{n\times m}} \|AX - I_m\|_F^2.$$

6. $\mathbf{0}_{mn}^\dagger = \mathbf{0}_{nm}$ and $J_{mn}^\dagger = \frac{1}{mn}J_{nm}$, where $\mathbf{0}_{mn} \in \mathbb{C}^{m\times n}$ is the all 0s matrix and $J_{mn} \in \mathbb{C}^{m\times n}$ is the all 1s matrix.
7. If $\mathbf{x} \neq \mathbf{0}$, $\mathbf{y} \neq \mathbf{0}$, then $(\mathbf{x}\mathbf{y}^*)^\dagger = \dfrac{\mathbf{y}\mathbf{x}^*}{\|\mathbf{x}\|^2\|\mathbf{y}\|^2}$.
8. If $\mathbf{x} \neq \mathbf{0}$, then $\mathbf{x}^\dagger = \dfrac{\mathbf{x}^*}{\|\mathbf{x}\|^2}$.
9. Let $\alpha$ be a scalar. Denote

$$\alpha^\dagger = \begin{cases} \alpha^{-1}, & \text{if } \alpha \neq 0, \\ 0, & \text{if } \alpha = 0. \end{cases}$$

Then

(a) $(\alpha A)^\dagger = \alpha^\dagger A^\dagger$.

(b) $(\operatorname{diag}(\beta_1, \beta_2, \cdots, \beta_n))^\dagger = \operatorname{diag}(\beta_1^\dagger, \beta_2^\dagger, \cdots, \beta_n^\dagger)$.

10. $(A^\dagger)^* = (A^*)^\dagger$; $(A^\dagger)^\dagger = A$.
11. If $A$ is a nonsingular square matrix, then $A^\dagger = A^{-1}$.
12. If $U$ has orthonormal columns or orthonormal rows, then $U^\dagger = U^*$.
13. If $A = A^*$ and $A = A^2$, then $A^\dagger = A$.
14. $A^\dagger = A^*$ if and only if $A^*A$ is idempotent.
15. If $A$ is normal and $k$ is a positive integer, then $AA^\dagger = A^\dagger A$ and $(A^k)^\dagger = (A^\dagger)^k$.
16. If $U \in \mathbb{C}^{m\times n}$ is of rank $n$ and satisfies $U^\dagger = U^*$, then $U$ has orthonormal columns.
17. If $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary matrices, then $(UAV)^\dagger = V^*A^\dagger U^*$.
18. $A^\dagger = (A^*A)^\dagger A^* = A^*(AA^*)^\dagger$. In particular,

(a) if $A \in \mathbb{C}^{m\times n}$ $(m \ge n)$ has full rank $n$, then $A^\dagger = (A^*A)^{-1}A^*$;

(b) if $A \in \mathbb{C}^{m\times n}$ $(m \le n)$ has full rank $m$, then $A^\dagger = A^*(AA^*)^{-1}$.

19. Let $A \in \mathbb{C}^{m\times n}$. Then



(a) $A^\dagger A$, $AA^\dagger$, $I_n - A^\dagger A$, and $I_m - AA^\dagger$ are orthogonal projections.

(b) $\operatorname{rank}(A) = \operatorname{rank}(A^\dagger) = \operatorname{rank}(AA^\dagger) = \operatorname{rank}(A^\dagger A)$.

(c) $\operatorname{rank}(I_n - A^\dagger A) = n - \operatorname{rank}(A)$.

(d) $\operatorname{rank}(I_m - AA^\dagger) = m - \operatorname{rank}(A)$.

20. $AA^\dagger = \operatorname{Proj}_{\operatorname{range}(A)}$; $A^\dagger A = \operatorname{Proj}_{\operatorname{range}(A^\dagger)}$.
21. Suppose that $A \in F^{m\times n}$, where $F = \mathbb{C}$ or $\mathbb{R}$. Then

(a) $\operatorname{range}(A) = \operatorname{range}(AA^*) = \operatorname{range}(AA^\dagger)$.

(b) $\operatorname{range}(A^\dagger) = \operatorname{range}(A^*) = \operatorname{range}(A^*A) = \operatorname{range}(A^\dagger A)$.

(c) $\ker(A) = \ker(A^*A) = \ker(A^\dagger A)$.

(d) $\ker(A^\dagger) = \ker(A^*) = \ker(AA^*) = \ker(AA^\dagger)$.

(e) $\operatorname{range}(A^\dagger A) \oplus \ker(A^\dagger A) = F^n$.

(f) $\operatorname{range}(AA^\dagger) \oplus \ker(AA^\dagger) = F^m$.

22. If $A = A_1 + A_2 + \cdots + A_k$, $A_i^*A_j = 0$, and $A_iA_j^* = 0$, for all $i, j = 1, \cdots, k$, $i \neq j$, then $A^\dagger = A_1^\dagger + A_2^\dagger + \cdots + A_k^\dagger$.
23. If $A$ is an $m\times r$ matrix of rank $r$ and $B$ is an $r\times n$ matrix of rank $r$, then $(AB)^\dagger = B^\dagger A^\dagger$.
24. $(A^*A)^\dagger = A^\dagger(A^*)^\dagger$; $(AA^*)^\dagger = (A^*)^\dagger A^\dagger$.
25. [Gre66] Each one of the following conditions is necessary and sufficient for $(AB)^\dagger = B^\dagger A^\dagger$:

(a) $\operatorname{range}(BB^*A^*) \subseteq \operatorname{range}(A^*)$ and $\operatorname{range}(A^*AB) \subseteq \operatorname{range}(B)$.

(b) $A^\dagger ABB^*$ and $A^*ABB^\dagger$ are both Hermitian matrices.

(c) $A^\dagger ABB^*A^* = BB^*A^*$ and $BB^\dagger A^*AB = A^*AB$.

(d) $A^\dagger ABB^*A^*ABB^\dagger = BB^*A^*A$.

(e) $A^\dagger AB = B(AB)^\dagger AB$ and $BB^\dagger A^* = A^*AB(AB)^\dagger$.

26. $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$, where $\otimes$ denotes the Kronecker product.
27. $A^\dagger = \lim\limits_{\alpha\to 0} A^*(\alpha I + AA^*)^{-1} = \lim\limits_{\alpha\to 0}(\alpha I + A^*A)^{-1}A^*$.
28. $A^\dagger = \sum\limits_{j=1}^{\infty} A^*(I + AA^*)^{-j} = \sum\limits_{j=1}^{\infty}(I + A^*A)^{-j}A^*$.
29. (Continuity of pseudo-inverse) Suppose that $A \in F^{m\times n}$ and $E \in F^{m\times n}$, where $F = \mathbb{C}$ or $\mathbb{R}$. Then $\lim\limits_{E\to 0}(A+E)^\dagger = A^\dagger$ if and only if there is $\epsilon > 0$ such that $\operatorname{rank}(A+E) = \operatorname{rank}(A)$ when $\|E\|_2 \le \epsilon$.
30. Let $A \in \mathbb{C}^{m\times n}$ be of rank $r$ where $0 < r < \min\{m,n\}$. Suppose that $A$ can be partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{11} \in \mathbb{C}^{r\times r}$ and $\operatorname{rank}(A_{11}) = r$. Then

$$A^\dagger = \begin{bmatrix} A_{11}^*XA_{11}^* & A_{11}^*XA_{21}^* \\ A_{12}^*XA_{11}^* & A_{12}^*XA_{21}^* \end{bmatrix},$$

where

$$X = (A_{11}A_{11}^* + A_{12}A_{12}^*)^{-1}A_{11}(A_{11}^*A_{11} + A_{21}^*A_{21})^{-1}.$$

# Reverse order law for the Moore–Penrose inverse ☆

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ABSTRACT

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## 1. Introduction

In this paper we extend some results from [15] to infinite-dimensional settings. Among other things, we obtain the reverse order law for the Moore–Penrose inverse as a corollary. We use the matrix form of a linear bounded operator, and this matrix form is induced by some natural decompositions of Hilbert spaces.

In the rest of the Introduction we formulate two auxiliary results. In Section 2 we present the results related to the reverse order rule for the Moore–Penrose inverse of Hilbert space operators with closed range. The present paper is the extension of results from [15] to infinite-dimensional settings.

## 2. Reverse order law

In this section we prove the results concerning the reverse order law for the Moore–Penrose inverse.

**Theorem 2.2.** *Let $X$, $Y$, $Z$ be Hilbert spaces, and let $A \in \mathcal{L}(Y,Z)$, $B \in \mathcal{L}(X,Y)$ be such that $A$, $B$, $AB$ have closed ranges. Then the following statements hold:*

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(c) *The following statements are equivalent:*
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 3. $A^*AB = BB^\dagger A^*AB$ *and* $ABB^* = ABB^*A^\dagger A$;
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**Proof.** The operators $A$ and $B$ have the same matrix representations as in the previous theorem. The following products will be useful:

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First, we find the equivalent expressions for our statements in terms of $A_1$, $A_2$ and $B_1$.



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 2. $A^*AB = BB^\dagger A^*AB \Leftrightarrow A_2^*A_1 = 0$.
 3. Notice that $\mathcal{R}(A^*AB) \subset \mathcal{R}(B)$ if and only if $BB^\dagger A^*AB = A^*AB$, so $2 \Leftrightarrow 3$.
 4. If we check properly the Penrose equations, then we see that: $B^\dagger A^\dagger \in (AB)\{1,2,3\} \Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1$ and $[A_1A_1^*, D^{-1}] = 0$.

Now, we prove the following: $1 \Leftrightarrow 2$, $4 \Rightarrow 2$ and $1 \Rightarrow 4$.

We prove $1 \Leftrightarrow 2$. Notice that

$$A_1B_1(A_1B_1)^\dagger = A_1A_1^*D^{-1} \quad \Leftrightarrow \quad (A_1B_1)^\dagger = (A_1B_1)^\dagger A_1A_1^*D^{-1}.$$

The last statement is obtained by multiplying the first expression by $(A_1B_1)^\dagger$ from the left side, or multiplying the second expression by $A_1B_1$ from the left side, and using $A_1A_1^* = A_1B_1B_1^{-1}A_1^*$. Now, there is a chain of the equivalences:

$$\begin{aligned} (A_1B_1)^\dagger = (A_1B_1)^\dagger A_1A_1^*D^{-1} \quad &\Leftrightarrow \quad (A_1B_1)^\dagger\big(A_1A_1^* + A_2A_2^*\big) = (A_1B_1)^\dagger A_1A_1^* \\ &\Leftrightarrow \quad (A_1B_1)^\dagger A_2A_2^* = 0 \quad \Leftrightarrow \quad \mathcal{R}\big(A_2A_2^*\big) \subset \mathcal{N}\big((A_1B_1)^\dagger\big) \\ &\Leftrightarrow \quad \mathcal{R}(A_2) \subset \mathcal{N}\big((A_1B_1)^*\big) \quad \Leftrightarrow \quad B_1^*A_1^*A_2 = 0 \quad \Leftrightarrow \quad A_1^*A_2 = 0. \end{aligned}$$

Therefore, we have just proved that $1 \Leftrightarrow 2$.

Now we prove $1 \Rightarrow 4$. If we multiply $A_1B_1(A_1B_1)^\dagger = A_1A_1^*D^{-1}$ by $A_1B_1$ from the right side, we get $A_1A_1^*D^{-1}A_1 = A_1$. Thus, 4 holds.

Finally, we prove $4 \Rightarrow 2$. If $A_1A_1^*D^{-1}A_1 = A_1$ and $[A_1A_1^*, D^{-1}] = 0$, then $A_1A_1^*A_1 = DA_1 = A_1A_1^*A_1 + A_2A_2^*A_1$, implying that $A_2A_2^*A_1 = 0$. Hence, $\mathcal{R}(A_1) \subset \mathcal{N}(A_2A_2^*) = \mathcal{N}(A_2^*)$, so $A_2^*A_1 = 0$. Thus, 2 holds.

Notice that the equivalence $3 \Leftrightarrow 4$ is proved in [8], also.

(b) 1. $(AB)^\dagger AB = B^\dagger A^\dagger AB \Leftrightarrow (A_1B_1)^\dagger A_1B_1 = B_1^{-1}A_1^*D^{-1}A_1B_1$. Moreover, $(A_1B_1)^\dagger A_1B_1$ is Hermitian, so $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$.
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 4. The Penrose equations imply that: $B^\dagger A^\dagger \in (AB)\{1,2,4\} \Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1$ and $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$.

We prove $1 \Rightarrow 4 \Rightarrow 2 \Rightarrow 1$.

Suppose that 1 holds. If we multiply $(A_1B_1)^\dagger A_1B_1 = B_1^{-1}A_1^*D^{-1}A_1B_1$ by $A_1B_1$ from the left side, we obtain $A_1 = A_1A_1^*D^{-1}A_1$. Furthermore, $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$ holds. Therefore, $1 \Rightarrow 4$.

Suppose that 4 holds. Obviously, $A_1B_1B_1^*A_1^*D^{-1}A_1 = A_1A_1^*D^{-1}A_1B_1B_1^* = A_1B_1B_1^*$. Thus, the first equality of 2 holds. The second equality of 2 also holds, since $A_1^*D^{-1}A_2 = 0 \Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1$, which is shown in the proof of Theorem 2.1. Here we use again $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$. Consequently, $4 \Rightarrow 2$.

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Finally, the part (c) follows from the parts (a) and (b). □

We also prove the following result.

**Theorem 2.3.** *Let $X$, $Y$, $Z$ be Hilbert spaces, and let $A \in \mathcal{L}(Y,Z)$, $B \in \mathcal{L}(X,Y)$ be such that $A$, $B$, $AB$ have closed ranges. Then we have:*

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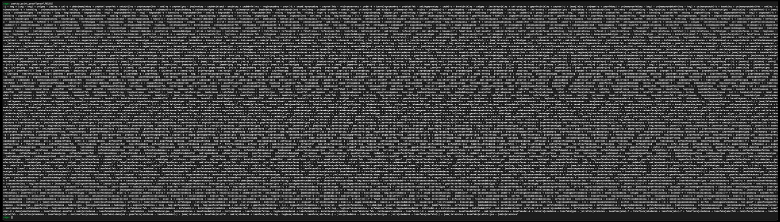

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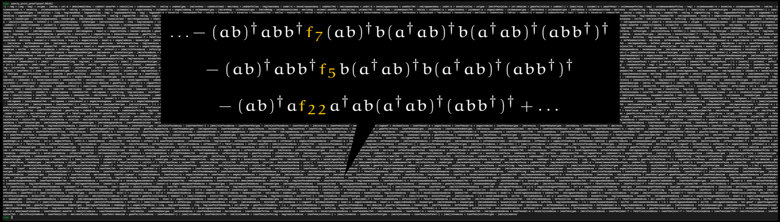

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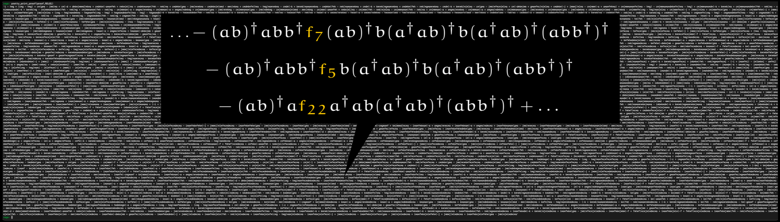


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& - f_{39}(a^\dagger)^*((ab)^\dagger)^*(ab)^\dagger + b^\dagger(abb^\dagger)^\dagger((abb^\dagger)^\dagger)^*(b^\dagger)^* f_{31} - b^\dagger f_{14} d^* b^*(a^\dagger)^* \\
& + (a^\dagger ab)^\dagger a^\dagger ab f_{12}(ab)^\dagger - b^\dagger(abb^\dagger)^\dagger f_{15}((ab)^\dagger)^* b^*(a^\dagger)^* \\
& + f_{20} b^*(a^\dagger)^*((ab)^\dagger)^*(ab)^\dagger + (a^\dagger ab)^\dagger a^\dagger abb^* f_{23}((ab)^\dagger)^*(ab)^\dagger
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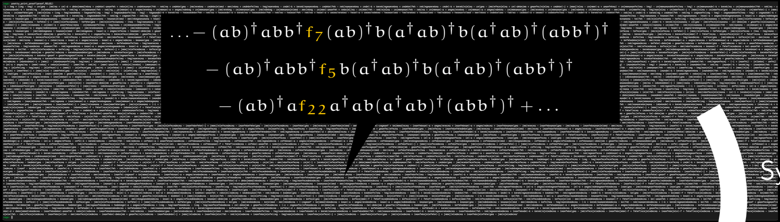


Syzygies + LP

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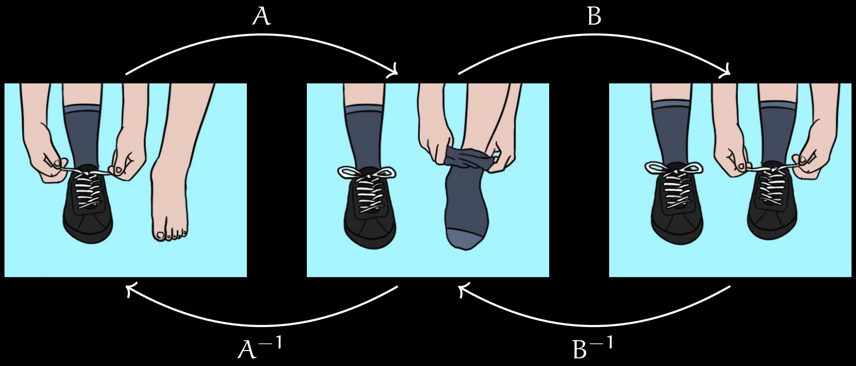
(ab)^\dagger - b^\dagger a^\dagger = {} & f_{21} - f_{10} + b^\dagger f_{14} - f_{12} (ab)^\dagger - b^\dagger (abb^\dagger)^\dagger f_{11} + b^\dagger (abb^\dagger)^\dagger f_{15} \\
& + (a^\dagger ab)^\dagger a^\dagger f_9 (ab)^\dagger - b^* f_{23} ((ab)^\dagger)^* (ab)^\dagger - f_{21} ab (ab)^\dagger + f_{22} ab (ab)^\dagger \\
& - f_{39} (a^\dagger)^* ((ab)^\dagger)^* (ab)^\dagger + b^\dagger (abb^\dagger)^\dagger ((abb^\dagger)^\dagger)^* (b^\dagger)^* f_{31} - b^\dagger f_{14} d^* b^* (a^\dagger)^* \\
& + (a^\dagger ab)^\dagger a^\dagger ab f_{12} (ab)^\dagger - b^\dagger (abb^\dagger)^\dagger f_{15} ((ab)^\dagger)^* b^* (a^\dagger)^* \\
& + f_{20} b^* (a^\dagger)^* ((ab)^\dagger)^* (ab)^\dagger + (a^\dagger ab)^\dagger a^\dagger abb^* f_{23} ((ab)^\dagger)^* (ab)^\dagger
\end{aligned}$$

# What about false statements?

$$\forall A, B, X, Y \;:\; (AX = 1 \;\wedge\; BY = 1) \;\Rightarrow\; ABXY = 1$$

# The Sock-Shoe-Principle

# The Sock-Shoe-Principle



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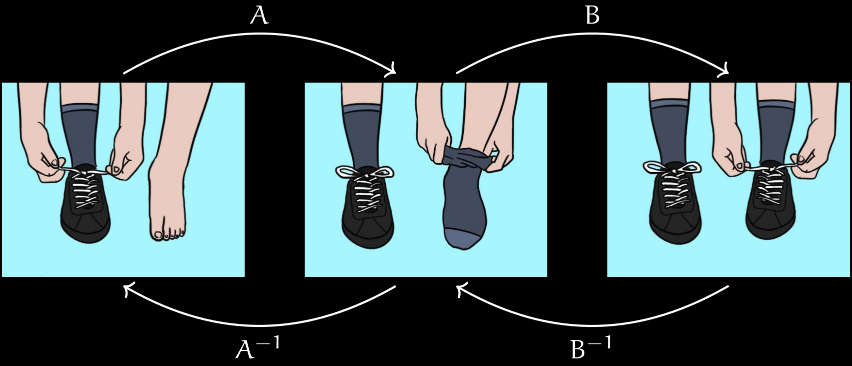


$$(AB)^{-1} = B^{-1}A^{-1}$$

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Plug in
and check

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Idea: make ansatz
with matrices
of fixed size

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \quad X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \quad Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$$

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SAT solving + Hensel lifting

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Does this always work? – No.

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Does this always work? – No.

Will a better method always work? – No.

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$$\Big\Downarrow$$

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SAT solving + Hensel lifting

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Does this always work? – No.

Will a better method always work? – No.

Does this work often enough? – Seems so.

# Satisfiability of polynomial system

| | |
|---|---|
| Given | $f_1, \ldots, f_r \in \mathbb{Z}[x_1, \ldots, x_n]$ |
| Compute | a common root of $f_1, \ldots, f_r$ over $\mathbb{Q}$ |

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Prop. formula
$(x \vee \neg y) \wedge (\neg x \vee y)$ $\longrightarrow$ SAT solver

$\longrightarrow$ ✔ $x \mapsto 1,\ y \mapsto 1$

✖

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Given a root over $\mathbb{Z}_{2^n}$, we can lift it to a root over $\mathbb{Z}_{2^{n+1}}$ by linear algebra (Hensel lifting).

Once we have a root over $\mathbb{Z}_{2^N}$ with $N$ large enough, we can perform rational reconstruction to recover a root over $\mathbb{Q}$.

# Counterexamples in practice

## Algebraic proof methods for identities of matrices and operators: improvements of Hartwig's triple reverse order law

Dragana S. Cvetković-Ilić[1], Clemens Hofstadler[2], Jamal Hossein Poor[2], Jovana Milošević[1], Clemens G. Raab[2], and Georg Regensburger[2]

[1]Department of Mathematics, Faculty of Sciences and Mathematics, University of Niš, Serbia
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**Theorem 2.1.** [*34*] *Let $A, B, C$ be complex matrices such that $ABC$ is defined and let $P = A^\dagger ABCC^\dagger$, $Q = CC^\dagger B^\dagger A^\dagger A$. The following conditions are equivalent:*

*(i)* $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$;

*(ii)* $Q \in P\{1, 2\}$ *and both of* $A^*APQ$ *and* $QPCC^*$ *are Hermitian;*

*(iii)* $Q \in P\{1, 2\}$ *and both of* $A^*APQ$ *and* $QPCC^*$ *are EP;*

*(iv)* $Q \in P\{1\}$, $\mathcal{R}(A^*AP) = \mathcal{R}(Q^*)$ *and* $\mathcal{R}(CC^*P^*) = \mathcal{R}(Q)$;

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$\subseteq$ $\supseteq$

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$\subseteq$

$\subseteq$ ?

**Example 2.5.** *Let*

$$A = \begin{bmatrix} -3 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad C = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

*Then*

$$A^\dagger = \frac{1}{17}\begin{bmatrix} -3 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \quad B^\dagger = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}, \quad C^\dagger = C.$$

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```
sage: from operator_gb import *
sage: F.<a,b,c,...> = FreeAlgebra(QQ)
sage: assumptions = [a*b*a - a,...]
sage: claim = abc_dag - c_dag*b_dag*a_dag
sage: counterexample(assumptions, claim)
```

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Noncommut.
Gröbner bases


Noncommut.
Gröbner bases
software
operator_gb
(SageMath)
NCAlgebra
(Mathematica)
Macaulay2
Opal (C++)
GBNP (GAP)
Letterplace
(Singular)
bergman
(Standard Lisp)
Oscar
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NCPoly
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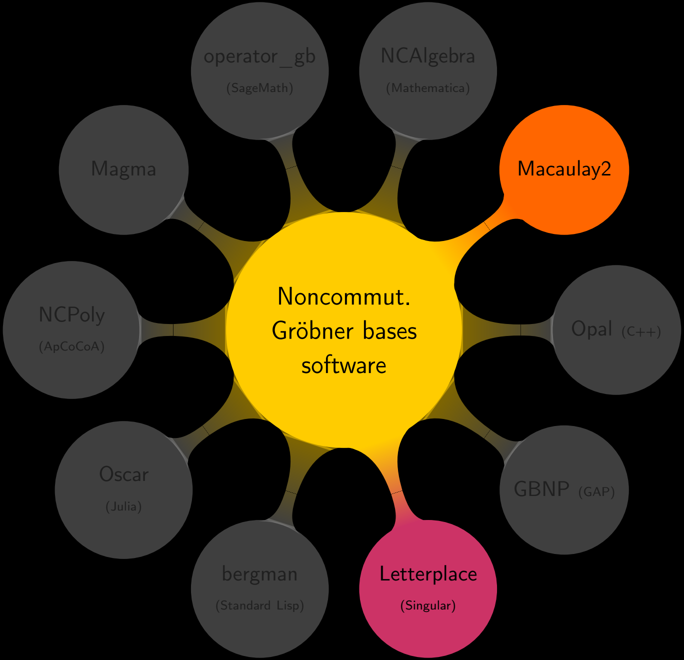
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| Example | Letterplace | Macaulay2 | f4ncgb | | |
|---|---|---|---|---|---|
| | | | 1 core | 4 cores | 16 cores |
| 4nilp5s-10 | 1282 | 875 | 150 | 79 | 63 |
| braid3-16 | 18 953 | 14 291 | 105 | 34 | 18 |
| braidX-18 | >43 200 | >43 200 | 1977 | 601 | 260 |
| braidXY-12 | 1847 | 18 887 | 62 | 52 | 52 |
| holt_G3562h-17 | >43 200 | >43 200 | 25 021 | 12 671 | 6824 |
| lascala_neuh-13 | 171 | 37 | 9 | 5 | 4 |
| lp1-15 | 24 166 | 33 923 | 266 | 179 | 155 |
| lv2d10-100 | >43 200 | 24 930 | 48 | 27 | 47 |
| malle_G12h-100 | 4142 | 163 | 89 | 74 | 73 |

(Timings in sec)


Noncommut. Gröbner bases software
operator_gb
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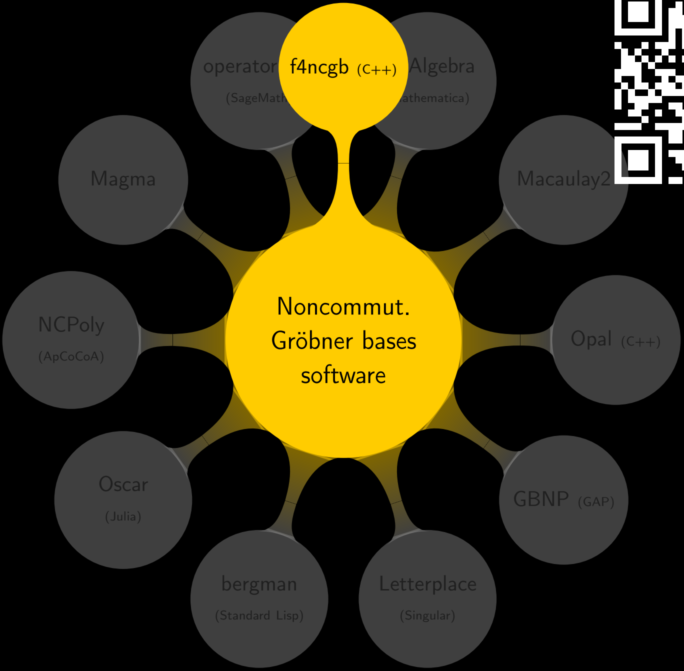
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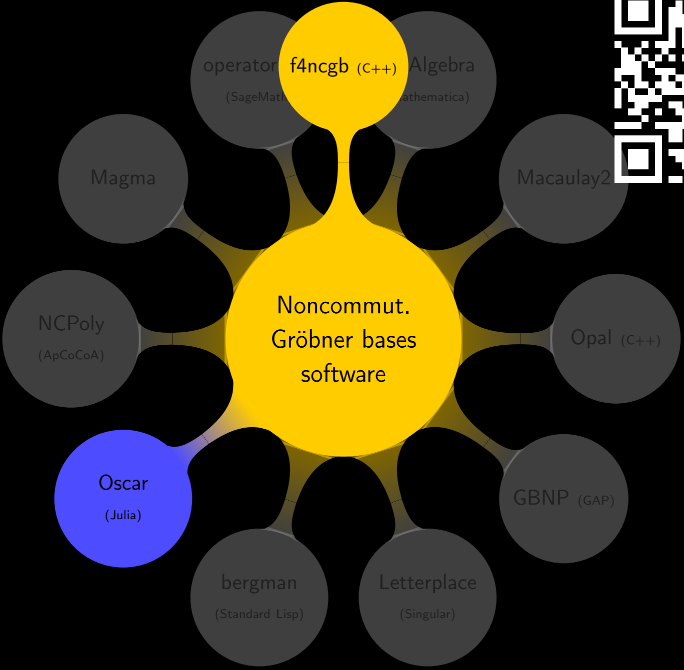
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# Algebraic First-Order Theorem Proving

$=$

Proving statements about linear operators
with computer algebra

| | | |
|---|---|---|
| Linear operators | $\longrightarrow$ | noncommutative polynomials in free algebra |
| Operator statement | $\longrightarrow$ | ideal membership $f \overset{?}{\in} I$ |
| Proof | $\longrightarrow$ | explicit linear combination |

We can also...

... compute short(est) proofs of true statements.

... compute counterexamples for false statements.

Hi ChatGPT

Hi there! 😊 What can I help you with today?

How would you prove or disprove a statement about linear operators?

I would use computer algebra.