Recent Advancements in Noncommutative Gröbner Basis Software

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based on joint work with Maximilian Heisinger

JOHANNES KEPLER UNIVERSITY LINZ

FWF Der Wissenschaftsfonds.

Noncommutative polynomials

noncommutative = really noncommutative

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Noncom. polynomial $\quad c_1 \cdot w_1 + \cdots + c_d \cdot w_d \quad \in K\langle x_1, \dots, x_n\rangle$

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words over $x_1, \ldots, x_n$

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$$\begin{aligned} \text{Multiplication} &= \text{concatenation of words} \\ (xy - 1) \cdot (yx + 2) &= xyyx + 2xy - yx - 2 \end{aligned}$$

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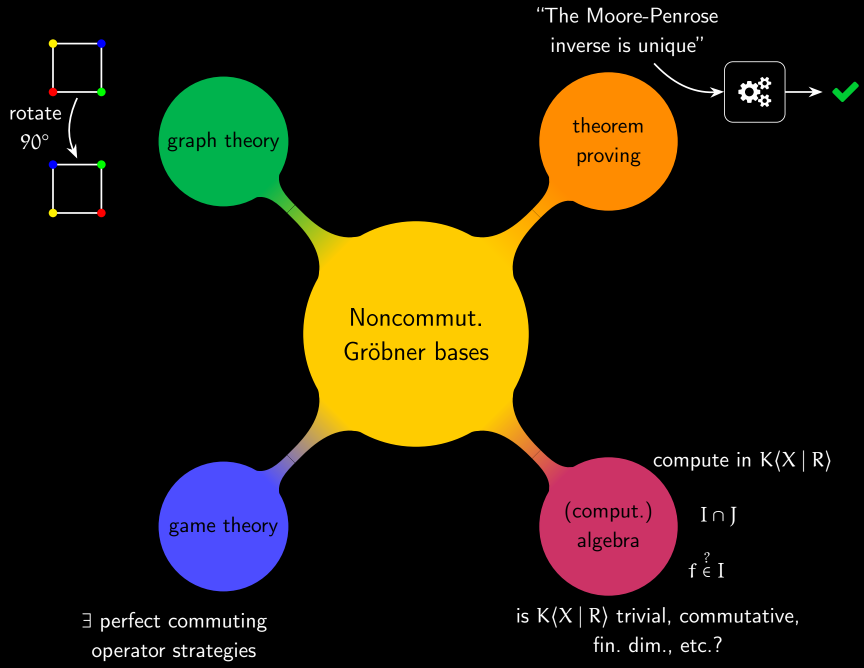
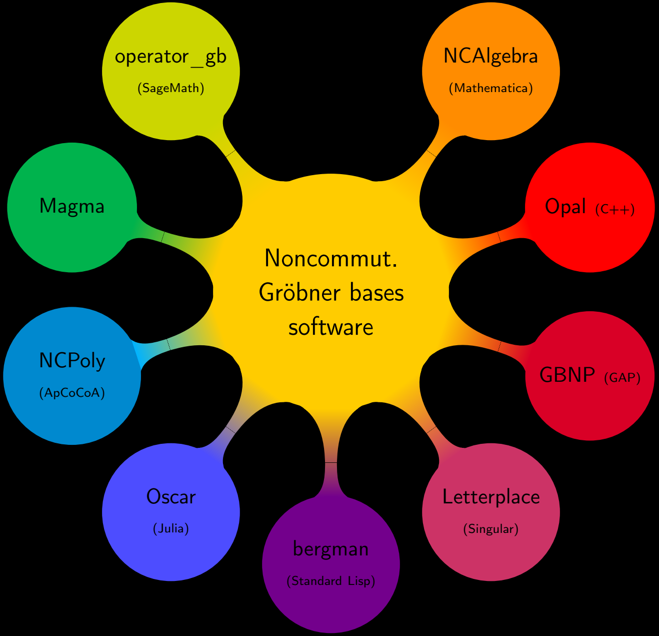
Noncomm. GB theory = comm. GB theory − finiteness



"The Moore-Penrose inverse is unique"
rotate 90°
graph theory
theorem proving
Noncommut. Gröbner bases
game theory
(comput.) algebra
compute in $K\langle X \mid R\rangle$
$I \cap J$
$f \overset{?}{\in} I$
$\exists$ perfect commuting operator strategies
is $K\langle X \mid R\rangle$ trivial, commutative, fin. dim., etc.?



Noncommut. Gröbner bases software
operator_gb (SageMath)
NCAlgebra (Mathematica)
Magma
Opal (C++)
NCPoly (ApCoCoA)
GBNP (GAP)
Oscar (Julia)
bergman (Standard Lisp)
Letterplace (Singular)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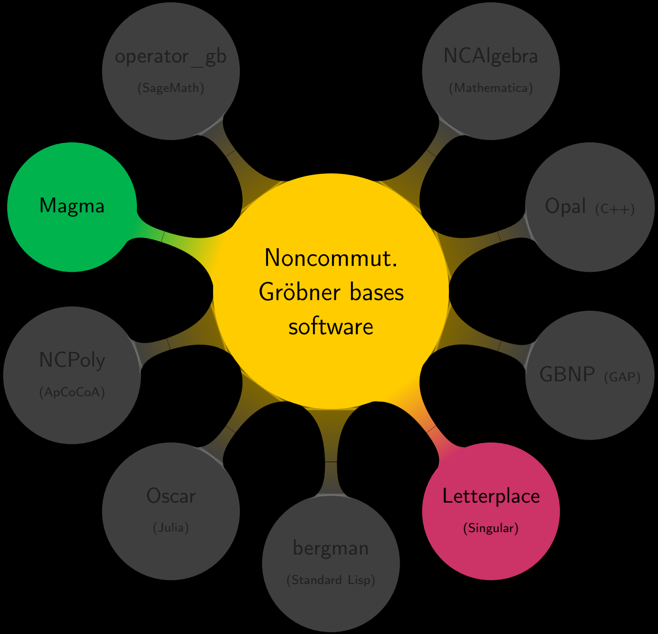
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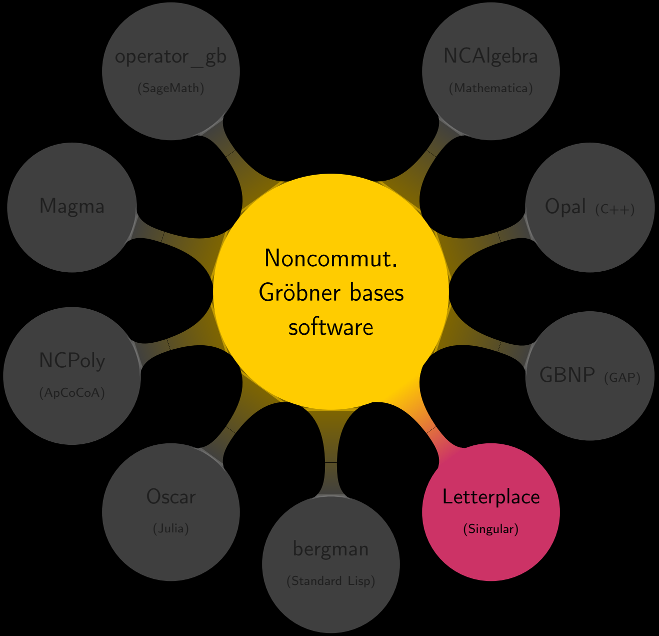
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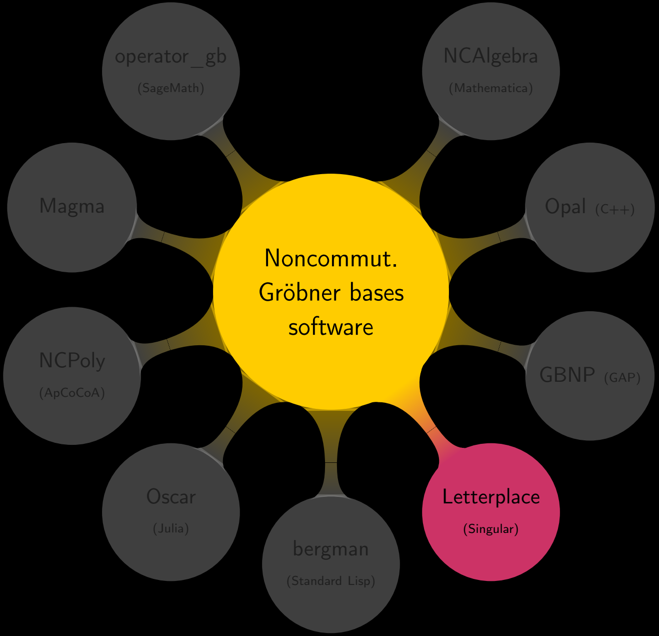
Example	Letterplace	f4ncgb 1 core	f4ncgb 4 cores	f4ncgb 16 cores
4nilp5s-10	1282	150	79	63
braid3-16	18 953	105	34	18
braidXY-12	1847	62	52	52
holt_G3562h-17	>43 200	25 021	12 671	6824
lascala_neuh-13	171	9	5	4
lp1-15	24 166	266	179	155
lv2d10-100	>43 200	48	27	47
malle_G12h-100	4142	89	74	73

(Timings in sec)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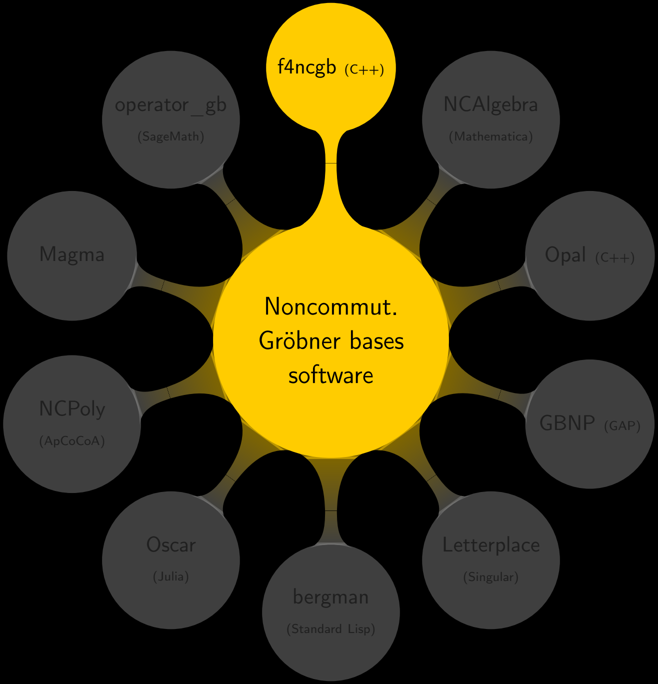
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f4ncgb

Open-source C++ library that ports commutative advancements to the noncommutative setting.

- Gröbner basis computation in $\mathbb{Q}\langle X\rangle$ and $\mathbb{Z}_p\langle X\rangle$ for prime $p < 2^{31}$
- Several orders of magnitude faster than current state of the art
- Proof logging via cofactor representations
- Soon part of OSCAR SYMBOLIC TOOLS

Data structures

- Monomials are unique and referenced
- Coefficients are shared
- Prefix tree for divisions

Algorithms

- Noncomm. F4 algorithm
- Sparse linear algebra (multi-modular, parallelized, probabilistic)
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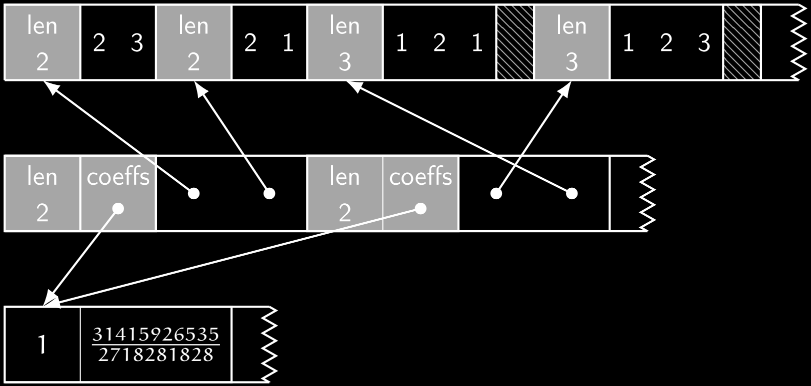
Monomials & Polynomials

Represent vars by index according to mon. order: $x_1 \prec x_2 \prec \cdots \prec x_n$

$\downarrow$ 1 $\downarrow$ 2 $\downarrow$ n



Monomial divisibility tests

Observation: divisor candidates are always the same (lm's of the GB)

Exploit this information $\rightsquigarrow$ keep prefix tree of all leading monomials

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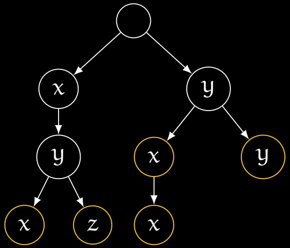
$$G = \{\, xyx - \ldots,\ xyz + \ldots,\ yx + \ldots,\ yxx - \ldots,\ yy + \ldots \}$$



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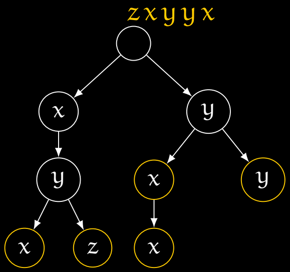

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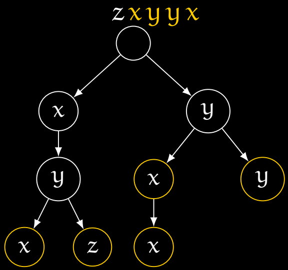

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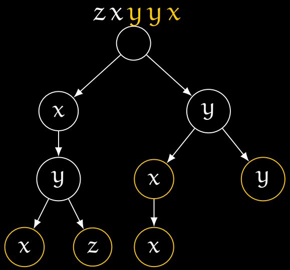

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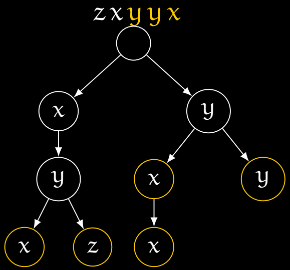

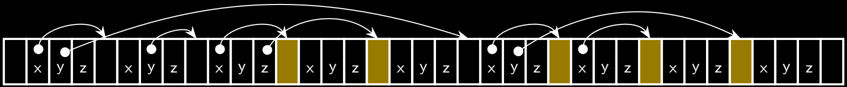

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Given input $f_1, \ldots, f_r$, write each $g \in GB$ as

$$g = \sum_j p_j \cdot f_j \cdot q_j$$

"*cofactor representation*"

with $p_j, q_j \in K\langle X \rangle$.

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$$\begin{pmatrix} — & p_1 & — \\ & \vdots & \\ — & p_k & — \end{pmatrix} \rightsquigarrow \quad \text{RRef}$$

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$$T \cdot \begin{pmatrix} — & p_1 & — \\ & \vdots & \\ — & p_k & — \end{pmatrix} = \text{RRef}$$

The rows of T give cofactor representations of g_i in terms of $f_1, \ldots, f_r$ and $g_1, \ldots, g_{i-1}$.

Substitution yields representations w.r.t. $f_1, \ldots, f_r$ (⚠ exp. blowup!)

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