First-order theorem proving for operator statements

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joint work with Georg Regensburger and Clemens G. Raab

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U N I KASSEL V E R S I T Ä T

Der Wissenschaftsfonds.

DISCRETE MATHEMATICS AND ITS APPLICATIONS
Series Editor KENNETH H. ROSEN

HANDBOOK OF LINEAR ALGEBRA

SECOND EDITION

$$\begin{bmatrix} 2 & 2 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Edited by
Leslie Hogben

CRC Press
Taylor & Francis Group
A CHAPMAN & HALL BOOK

5.7 Pseudo-Inverse

Definitions:

A **Moore–Penrose pseudo-inverse** of a matrix $A \in \mathbb{C}^{m\times n}$ is a matrix $A^\dagger \in \mathbb{C}^{n\times m}$ that satisfies the following four **Penrose** conditions:

$$AA^\dagger A = A; \quad A^\dagger AA^\dagger = A^\dagger; \quad (AA^\dagger)^* = AA^\dagger; \quad (A^\dagger A)^* = A^\dagger A.$$

Facts:

All the following facts except those with a specific reference can be found in [Gra83, pp. 105–141] or [RM71, pp. 44–67].

1. Every $A \in \mathbb{C}^{m\times n}$ has a unique pseudo-inverse $A^\dagger$.
2. If $A \in \mathbb{R}^{m\times n}$, then $A^\dagger$ is real.
3. If $A \in \mathbb{C}^{m\times n}$ of rank r has a full rank decomposition $A = BC$, where $B \in \mathbb{C}^{m\times r}$ and $C \in \mathbb{C}^{r\times n}$, then $A^\dagger$ can be evaluated using $A^\dagger = C^*(B^*AC^*)^{-1}B^*$.
4. [LH95, p. 38] If $A \in \mathbb{C}^{m\times n}$ of rank $r \le \min\{m,n\}$ has an SVD $A = U\Sigma V^*$, then its pseudo-inverse is $A^\dagger = V\Sigma^\dagger U^*$, where
$$\Sigma^\dagger = \operatorname{diag}(1/\sigma_1,\ldots,1/\sigma_r,0,\ldots,0) \in \mathbb{R}^{n\times m}.$$
5. [Hig96, p. 412] The pseudo-inverse $A^\dagger$ of $A \in F^{m\times n}$ ($F = \mathbb{C}$ or $\mathbb{R}$) solves the minimization problem
$$\min_{X\in F^{n\times m}} \|AX - I_m\|_F^2.$$
6. $\mathbf{0}_{mn}^\dagger = \mathbf{0}_{nm}$ and $J_{mn}^\dagger = \frac{1}{mn}J_{nm}$, where $\mathbf{0}_{mn} \in \mathbb{C}^{m\times n}$ is the all 0s matrix and $J_{mn} \in \mathbb{C}^{m\times n}$ is the all 1s matrix.
7. If $\mathbf{x} \ne \mathbf{0}$, $\mathbf{y} \ne \mathbf{0}$, then $(\mathbf{x}\mathbf{y}^*)^\dagger = \dfrac{\mathbf{y}\mathbf{x}^*}{\|\mathbf{x}\|^2\|\mathbf{y}\|^2}$.
8. If $\mathbf{x} \ne \mathbf{0}$, then $\mathbf{x}^\dagger = \dfrac{\mathbf{x}^*}{\|\mathbf{x}\|^2}$.
9. Let α be a scalar. Denote
$$\alpha^\dagger = \begin{cases}\alpha^{-1}, & \text{if } \alpha \ne 0,\\ 0, & \text{if } \alpha = 0.\end{cases}$$
Then
   (a) $(\alpha A)^\dagger = \alpha^\dagger A^\dagger$.
   (b) $(\operatorname{diag}(\beta_1,\beta_2,\cdots,\beta_n))^\dagger = \operatorname{diag}(\beta_1^\dagger,\beta_2^\dagger,\cdots,\beta_n^\dagger)$.
10. $(A^\dagger)^* = (A^*)^\dagger$; $(A^\dagger)^\dagger = A$.
11. If A is a nonsingular square matrix, then $A^\dagger = A^{-1}$.
12. If U has orthonormal columns or orthonormal rows, then $U^\dagger = U^*$.
13. If $A = A^*$ and $A = A^2$, then $A^\dagger = A$.
14. $A^\dagger = A^*$ if and only if A^*A is idempotent.
15. If A is normal and k is a positive integer, then $AA^\dagger = A^\dagger A$ and $(A^k)^\dagger = (A^\dagger)^k$.
16. If $U \in \mathbb{C}^{m\times n}$ is of rank n and satisfies $U^\dagger = U^*$, then U has orthonormal columns.
17. If $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary matrices, then $(UAV)^\dagger = V^*A^\dagger U^*$.
18. $A^\dagger = (A^*A)^\dagger A^* = A^*(AA^*)^\dagger$. In particular,
   (a) if $A \in \mathbb{C}^{m\times n}$ $(m \ge n)$ has full rank n, then $A^\dagger = (A^*A)^{-1}A^*$;
   (b) if $A \in \mathbb{C}^{m\times n}$ $(m \le n)$ has full rank m, then $A^\dagger = A^*(AA^*)^{-1}$.
19. Let $A \in \mathbb{C}^{m\times n}$. Then

   (a) $A^\dagger A$, $AA^\dagger$, $I_n - A^\dagger A$, and $I_m - AA^\dagger$ are orthogonal projections.
   (b) $\operatorname{rank}(A) = \operatorname{rank}(A^\dagger) = \operatorname{rank}(AA^\dagger) = \operatorname{rank}(A^\dagger A)$.
   (c) $\operatorname{rank}(I_n - A^\dagger A) = n - \operatorname{rank}(A)$.
   (d) $\operatorname{rank}(I_m - AA^\dagger) = m - \operatorname{rank}(A)$.
20. $AA^\dagger = \operatorname{Proj}_{\operatorname{range}(A)}$; $A^\dagger A = \operatorname{Proj}_{\operatorname{range}(A^\dagger)}$.
21. Suppose that $A \in F^{m\times n}$, where $F = \mathbb{C}$ or $\mathbb{R}$. Then
   (a) $\operatorname{range}(A) = \operatorname{range}(AA^*) = \operatorname{range}(AA^\dagger)$.
   (b) $\operatorname{range}(A^\dagger) = \operatorname{range}(A^*) = \operatorname{range}(A^*A) = \operatorname{range}(A^\dagger A)$.
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   (d) $\ker(A^\dagger) = \ker(A^*) = \ker(AA^*) = \ker(AA^\dagger)$.
   (e) $\operatorname{range}(A^\dagger A) \oplus \ker(A^\dagger A) = F^n$.
   (f) $\operatorname{range}(AA^\dagger) \oplus \ker(AA^\dagger) = F^m$.
22. If $A = A_1 + A_2 + \cdots + A_k$, $A_i^*A_j = 0$, and $A_iA_j^* = 0$, for all $i,j = 1,\cdots,k$, $i \ne j$, then $A^\dagger = A_1^\dagger + A_2^\dagger + \cdots + A_k^\dagger$.
23. If A is an $m\times r$ matrix of rank r and B is an $r\times n$ matrix of rank r, then $(AB)^\dagger = B^\dagger A^\dagger$.
24. $(A^*A)^\dagger = A^\dagger(A^*)^\dagger$; $(AA^*)^\dagger = (A^*)^\dagger A^\dagger$.
25. [Gre66] Each one of the following conditions is necessary and sufficient for $(AB)^\dagger = B^\dagger A^\dagger$:
   (a) $\operatorname{range}(BB^*A^*) \subseteq \operatorname{range}(A^*)$ and $\operatorname{range}(A^*AB) \subseteq \operatorname{range}(B)$.
   (b) $A^\dagger ABB^*$ and $A^*ABB^\dagger$ are both Hermitian matrices.
   (c) $A^\dagger ABB^*A^* = BB^*A^*$ and $BB^\dagger A^*AB = A^*AB$.
   (d) $A^\dagger ABB^*A^*ABB^\dagger = BB^*A^*A$.
   (e) $A^\dagger AB = B(AB)^\dagger AB$ and $BB^\dagger A^* = A^*AB(AB)^\dagger$.
26. $(A\otimes B)^\dagger = A^\dagger \otimes B^\dagger$, where $\otimes$ denotes the Kronecker product.
27. $A^\dagger = \lim\limits_{\alpha\to 0} A^*(\alpha I + AA^*)^{-1} = \lim\limits_{\alpha\to 0}(\alpha I + A^*A)^{-1}A^*$.
28. $A^\dagger = \sum\limits_{j=1}^{\infty} A^*(I + AA^*)^{-j} = \sum\limits_{j=1}^{\infty}(I + A^*A)^{-j}A^*$.
29. (Continuity of pseudo-inverse) Suppose that $A \in F^{m\times n}$ and $E \in F^{m\times n}$, where $F = \mathbb{C}$ or $\mathbb{R}$. Then $\lim\limits_{E\to 0}(A+E)^\dagger = A^\dagger$ if and only if there is $\epsilon > 0$ such that $\operatorname{rank}(A+E) = \operatorname{rank}(A)$ when $\|E\|_2 \le \epsilon$.
30. Let $A \in \mathbb{C}^{m\times n}$ be of rank r where $0 < r < \min\{m,n\}$. Suppose that A can be partitioned as
$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix},$$
where $A_{11} \in \mathbb{C}^{r\times r}$ and $\operatorname{rank}(A_{11}) = r$. Then
$$A^\dagger = \begin{bmatrix} A_{11}^*XA_{11}^* & A_{11}^*XA_{21}^* \\ A_{12}^*XA_{11}^* & A_{12}^*XA_{21}^*\end{bmatrix},$$
where
$$X = (A_{11}A_{11}^* + A_{12}A_{12}^*)^{-1}A_{11}(A_{11}^*A_{11} + A_{21}^*A_{21})^{-1}.$$

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- Correctness of first-order operator statements
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Software

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Automated proofs of operator statements

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Multiplication $=$ Concatenation of words

$$(xy - 1) \cdot (yx + 2) \quad = \quad xyyx + 2xy - yx - 2$$

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Fact: $f \in (f_1, \ldots, f_r) \iff \exists\, p_{i,j}, q_{i,j} \;:\; f = \sum_{i,j} p_{i,j} \cdot f_i \cdot q_{i,j}$

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If such a proof exists, it can be computed using noncom. Gröbner bases.

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  - if $f \notin (f_1, \ldots, f_r)$, we might run into an infinite computation

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$$B = \ldots = C \quad \Longleftrightarrow \quad ?$$

"The Moore-Penrose inverse is unique"

Def.: A matrix B is a Moore-Penrose inverse of a matrix A if

$$aba - a, \qquad bab - b, \qquad b^*a^* - ab, \qquad a^*b^* - ba$$

Claim If B and C satisfy these identities, then $B = C$.

Proof $$B = BAB = BACAB = \ldots = C$$

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$$L = R \quad \Longleftrightarrow \quad l - r \in \mathbb{Z}\langle \mathbf{X} \rangle$$
$$B = \ldots = C \quad \Longleftrightarrow \quad b - c \in (f_1, \ldots, f_{12})$$

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Proof Using our software package operator_gb...

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```

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Observation Proof only relies on basic linearity properties

$\Rightarrow$ Statement proven for matrices, (un)bounded operators, morphisms,...

Operator statements

Operators

- $0, A, B, C, \ldots$
- $S + T, \;\; S \cdot T, \;\; f(T_1, \ldots, T_n)$

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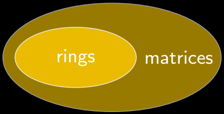

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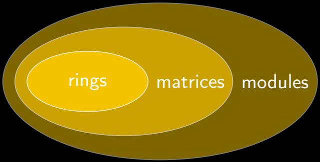

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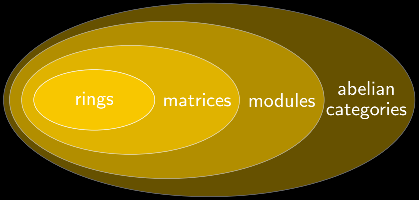

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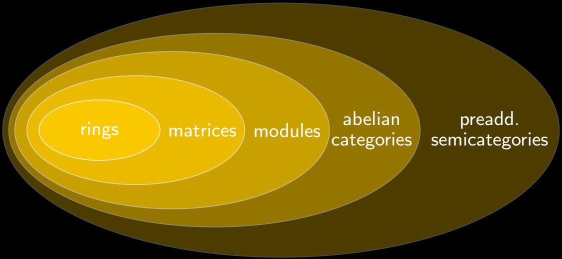

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Operator statements

$$S = T, \quad \neg\, \varphi, \quad (\varphi \wedge \psi), \quad (\varphi \vee \psi), \quad (\varphi \Rightarrow \psi), \quad \exists X : \varphi, \quad \forall X : \varphi$$

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Definition An operator statement is universally true if it follows from linearity.

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Definition An operator statement is universally true if it follows from linearity.

Fact: Determining universal truth is not decidable

Best we can hope for: semi-decision procedure

Determining universal truth

Quasi-identities

(Helton, Stankus, Wavrik '98, Schmitz, Levandovskyy '20, Raab, Regensburger, Hossein Poor '21)

$$\forall \mathbf{X} : \bigwedge_{i=1}^{m} P_i = Q_i \;\Rightarrow\; S = T \qquad \text{iff} \qquad s - t \in \left(p_1 - q_1, \ldots, p_m - q_m\right)$$

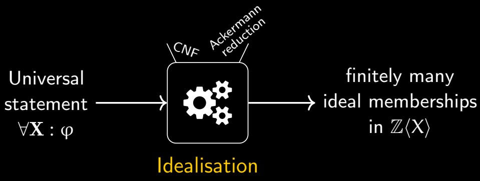
Determining universal truth

Universal statements

Universal
statement
$\forall \mathbf{X} : \varphi$

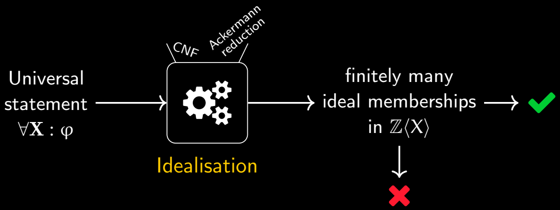
Determining universal truth

Universal statements



Determining universal truth

Universal statements



Theorem (H., Raab, Regensburger '22)
A universal statement is universally true iff its idealisation is true

Determining universal truth

Universal statements

verifiable using computer algebra

CNF

Ackermann reduction

Universal statement $\forall \mathbf{X} : \varphi$ → Idealisation → finitely many ideal memberships in $\mathbb{Z}\langle X\rangle$ →

Theorem (H., Raab, Regensburger '22)
A universal statement is universally true iff its idealisation is true

5.7 Pseudo-Inverse

Definitions:

A **Moore–Penrose pseudo-inverse** of a matrix $A \in \mathbb{C}^{m\times n}$ is a matrix $A^\dagger \in \mathbb{C}^{n\times m}$ that satisfies the following four **Penrose** conditions:

$$AA^\dagger A = A; \quad A^\dagger AA^\dagger = A^\dagger; \quad (AA^\dagger)^* = AA^\dagger; \quad (A^\dagger A)^* = A^\dagger A.$$

Facts:

All the following facts except those with a specific reference can be found in [Gra83, pp. 105–141] or [RM71, pp. 44–67].

1. Every $A \in \mathbb{C}^{m\times n}$ has a unique pseudo-inverse $A^\dagger$.
2. If $A \in \mathbb{R}^{m\times n}$, then $A^\dagger$ is real.
3. If $A \in \mathbb{C}^{m\times n}$ of rank r has a full rank decomposition $A = BC$, where $B \in \mathbb{C}^{m\times r}$ and $C \in \mathbb{C}^{r\times n}$, then $A^\dagger$ can be evaluated using $A^\dagger = C^*(B^*AC^*)^{-1}B^*$.
4. [LH95, p. 38] If $A \in \mathbb{C}^{m\times n}$ of rank $r \le \min\{m,n\}$ has an SVD $A = U\Sigma V^*$, then its pseudo-inverse is $A^\dagger = V\Sigma^\dagger U^*$, where
$$\Sigma^\dagger = \operatorname{diag}(1/\sigma_1,\ldots,1/\sigma_r,0,\ldots,0) \in \mathbb{R}^{n\times m}.$$
5. [Hig96, p. 412] The pseudo-inverse $A^\dagger$ of $A \in F^{m\times n}$ ($F = \mathbb{C}$ or $\mathbb{R}$) solves the minimization problem
$$\min_{X\in F^{n\times m}} \|AX - I_m\|_F^2.$$
6. $\mathbf{0}_{mn}^\dagger = \mathbf{0}_{nm}$ and $J_{mn}^\dagger = \frac{1}{mn}J_{nm}$, where $\mathbf{0}_{mn} \in \mathbb{C}^{m\times n}$ is the all 0s matrix and $J_{mn} \in \mathbb{C}^{m\times n}$ is the all 1s matrix.
7. If $\mathbf{x} \ne \mathbf{0}$, $\mathbf{y} \ne \mathbf{0}$, then $(\mathbf{x}\mathbf{y}^*)^\dagger = \dfrac{\mathbf{y}\mathbf{x}^*}{\|\mathbf{x}\|^2\|\mathbf{y}\|^2}$.
8. If $\mathbf{x} \ne \mathbf{0}$, then $\mathbf{x}^\dagger = \dfrac{\mathbf{x}^*}{\|\mathbf{x}\|^2}$.
9. Let α be a scalar. Denote
$$\alpha^\dagger = \begin{cases}\alpha^{-1}, & \text{if } \alpha \ne 0,\\ 0, & \text{if } \alpha = 0.\end{cases}$$
Then
   (a) $(\alpha A)^\dagger = \alpha^\dagger A^\dagger$.
   (b) $(\operatorname{diag}(\beta_1,\beta_2,\cdots,\beta_n))^\dagger = \operatorname{diag}(\beta_1^\dagger,\beta_2^\dagger,\cdots,\beta_n^\dagger)$.
10. $(A^\dagger)^* = (A^*)^\dagger$; $(A^\dagger)^\dagger = A$.
11. If A is a nonsingular square matrix, then $A^\dagger = A^{-1}$.
12. If U has orthonormal columns or orthonormal rows, then $U^\dagger = U^*$.
13. If $A = A^*$ and $A = A^2$, then $A^\dagger = A$.
14. $A^\dagger = A^*$ if and only if A^*A is idempotent.
15. If A is normal and k is a positive integer, then $AA^\dagger = A^\dagger A$ and $(A^k)^\dagger = (A^\dagger)^k$.
16. If $U \in \mathbb{C}^{m\times n}$ is of rank n and satisfies $U^\dagger = U^*$, then U has orthonormal columns.
17. If $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary matrices, then $(UAV)^\dagger = V^*A^\dagger U^*$.
18. $A^\dagger = (A^*A)^\dagger A^* = A^*(AA^*)^\dagger$. In particular,
   (a) if $A \in \mathbb{C}^{m\times n}$ ($m \ge n$) has full rank n, then $A^\dagger = (A^*A)^{-1}A^*$;
   (b) if $A \in \mathbb{C}^{m\times n}$ ($m \le n$) has full rank m, then $A^\dagger = A^*(AA^*)^{-1}$.
19. Let $A \in \mathbb{C}^{m\times n}$. Then
   (a) $A^\dagger A$, $AA^\dagger$, $I_n - A^\dagger A$, and $I_m - AA^\dagger$ are orthogonal projections.
   (b) $\operatorname{rank}(A) = \operatorname{rank}(A^\dagger) = \operatorname{rank}(AA^\dagger) = \operatorname{rank}(A^\dagger A)$.
   (c) $\operatorname{rank}(I_n - A^\dagger A) = n - \operatorname{rank}(A)$.
   (d) $\operatorname{rank}(I_m - AA^\dagger) = m - \operatorname{rank}(A)$.
20. $AA^\dagger = \operatorname{Proj}_{\operatorname{range}(A)}$; $A^\dagger A = \operatorname{Proj}_{\operatorname{range}(A^\dagger)}$.
21. Suppose that $A \in F^{m\times n}$, where $F = \mathbb{C}$ or $\mathbb{R}$. Then
   (a) $\operatorname{range}(A) = \operatorname{range}(AA^*) = \operatorname{range}(AA^\dagger)$.
   (b) $\operatorname{range}(A^\dagger) = \operatorname{range}(A^*) = \operatorname{range}(A^*A) = \operatorname{range}(A^\dagger A)$.
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   (d) $\ker(A^\dagger) = \ker(A^*) = \ker(AA^*) = \ker(AA^\dagger)$.
   (e) $\operatorname{range}(A^\dagger A) \oplus \ker(A^\dagger A) = F^n$.
   (f) $\operatorname{range}(AA^\dagger) \oplus \ker(AA^\dagger) = F^m$.
22. If $A = A_1 + A_2 + \cdots + A_k$, $A_i^*A_j = 0$, and $A_iA_j^* = 0$, for all $i,j = 1,\cdots,k$, $i \ne j$, then $A^\dagger = A_1^\dagger + A_2^\dagger + \cdots + A_k^\dagger$.
23. If A is an $m\times r$ matrix of rank r and B is an $r\times n$ matrix of rank r, then $(AB)^\dagger = B^\dagger A^\dagger$.
24. $(A^*A)^\dagger = A^\dagger(A^*)^\dagger$; $(AA^*)^\dagger = (A^*)^\dagger A^\dagger$.
25. [Gre66] Each one of the following conditions is necessary and sufficient for $(AB)^\dagger = B^\dagger A^\dagger$:
   (a) $\operatorname{range}(BB^*A^*) \subseteq \operatorname{range}(A^*)$ and $\operatorname{range}(A^*AB) \subseteq \operatorname{range}(B)$.
   (b) $A^\dagger ABB^*$ and $A^*ABB^\dagger$ are both Hermitian matrices.
   (c) $A^\dagger ABB^*A^* = BB^*A^*$ and $BB^\dagger A^*AB = A^*AB$.
   (d) $A^\dagger ABB^*A^*ABB^\dagger = BB^*A^*A$.
   (e) $A^\dagger AB = B(AB)^\dagger AB$ and $BB^\dagger A^* = A^*AB(AB)^\dagger$.
26. $(A\otimes B)^\dagger = A^\dagger \otimes B^\dagger$, where $\otimes$ denotes the Kronecker product.
27. $A^\dagger = \lim_{\alpha\to 0} A^*(\alpha I + AA^*)^{-1} = \lim_{\alpha\to 0}(\alpha I + A^*A)^{-1}A^*$.
28. $A^\dagger = \sum_{j=1}^{\infty} A^*(I + AA^*)^{-j} = \sum_{j=1}^{\infty}(I + A^*A)^{-j}A^*$.
29. (Continuity of pseudo-inverse) Suppose that $A \in F^{m\times n}$ and $E \in F^{m\times n}$, where $F = \mathbb{C}$ or $\mathbb{R}$. Then $\lim_{E\to 0}(A+E)^\dagger = A^\dagger$ if and only if there is $\epsilon > 0$ such that $\operatorname{rank}(A+E) = \operatorname{rank}(A)$ when $\|E\|_2 \le \epsilon$.
30. Let $A \in \mathbb{C}^{m\times n}$ be of rank r where $0 < r < \min\{m,n\}$. Suppose that A can be partitioned as
$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix},$$
where $A_{11} \in \mathbb{C}^{r\times r}$ and $\operatorname{rank}(A_{11}) = r$. Then
$$A^\dagger = \begin{bmatrix} A_{11}^*XA_{11}^* & A_{11}^*XA_{21}^* \\ A_{12}^*XA_{11}^* & A_{12}^*XA_{21}^*\end{bmatrix},$$
where
$$X = (A_{11}A_{11}^* + A_{12}A_{12}^*)^{-1}A_{11}(A_{11}^*A_{11} + A_{21}^*A_{21})^{-1}.$$

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Claim $\quad \exists X : (PA^*A = A \wedge AA^*Q = A) \;\Rightarrow\; \mathrm{mp}(A, X)$

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2. Plug in the explicit expression $\rightsquigarrow$ removes existential quantifier
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```

```
[x - a_adj*q*x, x - a_adj*p*x,
            x - a_adj*q*p_adj, x - a_adj*x_adj*x]
```

$\Rightarrow$ $X = A^*QP^*$ is MP-inverse of A

(can be certified using the software)

Existential statements

In the previous example, we found a suitable polynomial expression.

Question Was this just luck?

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Existential statements

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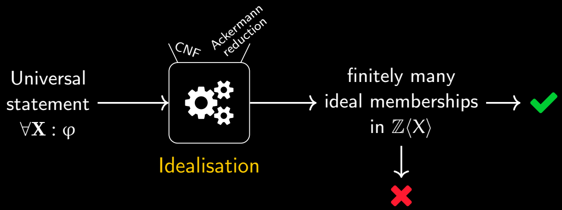
Such expressions always exist and the possible candidates are enumerable.

- Enumerating all possible expressions is hopeless
- Requires good heuristics $\longrightarrow$ provided by computer algebra
- Several heuristics implemented in `operator_gb`
  (ansatz, variable elimination, Gröbner basis techniques,...)

Determining universal truth

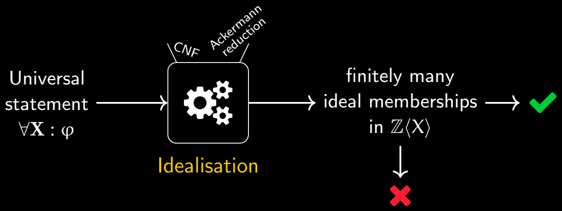
Universal statements



Theorem (H., Raab, Regensburger '22)
A universal statement is universally true iff its idealisation is true

Determining universal truth

General operator statements



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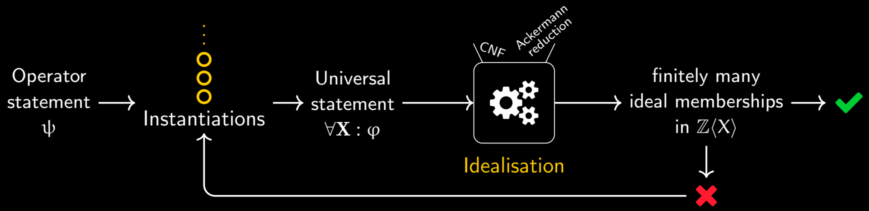

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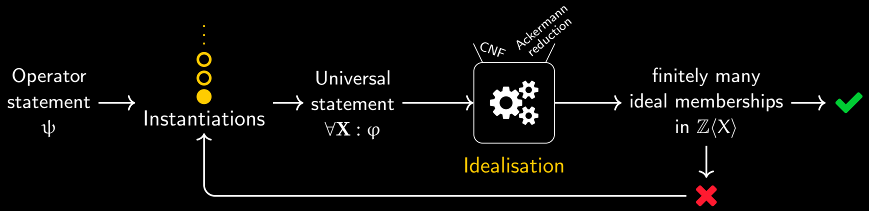

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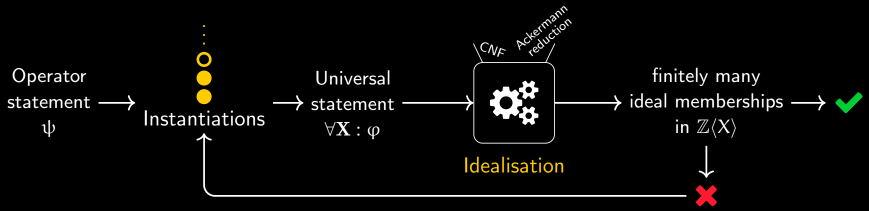

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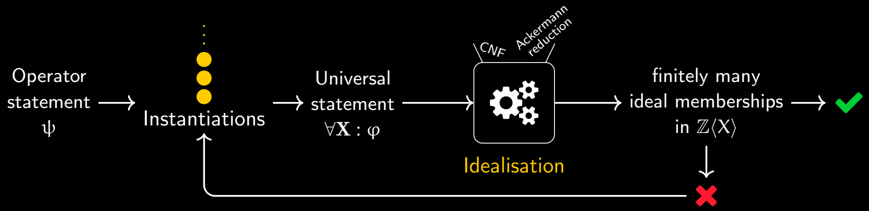

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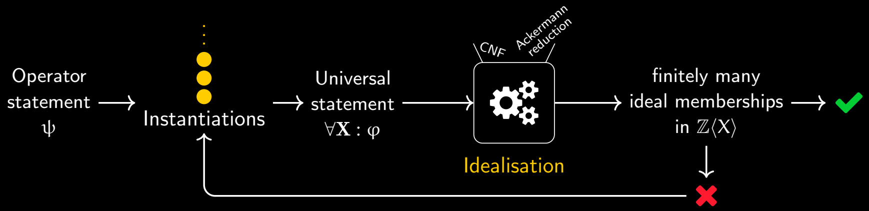

Theorem (H., Raab, Regensburger '22)

An operator statement is universally true iff the procedure terminates and returns ✓

5.7 Pseudo-Inverse

Definitions:

A **Moore–Penrose pseudo-inverse** of a matrix $A \in \mathbb{C}^{m\times n}$ is a matrix $A^\dagger \in \mathbb{C}^{n\times m}$ that satisfies the following four **Penrose** conditions:

$$AA^\dagger A = A; \quad A^\dagger AA^\dagger = A^\dagger; \quad (AA^\dagger)^* = AA^\dagger; \quad (A^\dagger A)^* = A^\dagger A.$$

Facts:

All the following facts except those with a specific reference can be found in [Gra83, pp. 105–141] or [RM71, pp. 44–67].

1. Every $A \in \mathbb{C}^{m\times n}$ has a unique pseudo-inverse $A^\dagger$.
2. If $A \in \mathbb{R}^{m\times n}$, then $A^\dagger$ is real.
3. If $A \in \mathbb{C}^{m\times n}$ of rank r has a full rank decomposition $A = BC$, where $B \in \mathbb{C}^{m\times r}$ and $C \in \mathbb{C}^{r\times n}$, then $A^\dagger$ can be evaluated using $A^\dagger = C^*(B^*AC^*)^{-1}B^*$.
4. [LH95, p. 38] If $A \in \mathbb{C}^{m\times n}$ of rank $r \le \min\{m,n\}$ has an SVD $A = U\Sigma V^*$, then its pseudo-inverse is $A^\dagger = V\Sigma^\dagger U^*$, where
$$\Sigma^\dagger = \mathrm{diag}(1/\sigma_1, \ldots, 1/\sigma_r, 0, \ldots, 0) \in \mathbb{R}^{n\times m}.$$
5. [Hig96, p. 412] The pseudo-inverse $A^\dagger$ of $A \in F^{m\times n}$ ($F = \mathbb{C}$ or $\mathbb{R}$) solves the minimization problem
$$\min_{X\in F^{n\times m}} \|AX - I_m\|_F^2.$$
6. $\mathbf{0}_{mn}^\dagger = \mathbf{0}_{nm}$ and $J_{mn}^\dagger = \frac{1}{mn}J_{nm}$, where $\mathbf{0}_{mn} \in \mathbb{C}^{m\times n}$ is the all 0s matrix and $J_{mn} \in \mathbb{C}^{m\times n}$ is the all 1s matrix.
7. If $\mathbf{x} \ne \mathbf{0}$, $\mathbf{y} \ne \mathbf{0}$, then $(\mathbf{x}\mathbf{y}^*)^\dagger = \dfrac{\mathbf{y}\mathbf{x}^*}{\|\mathbf{x}\|^2\|\mathbf{y}\|^2}$.
8. If $\mathbf{x} \ne \mathbf{0}$, then $\mathbf{x}^\dagger = \dfrac{\mathbf{x}^*}{\|\mathbf{x}\|^2}$.
9. Let α be a scalar. Denote
$$\alpha^\dagger = \begin{cases} \alpha^{-1}, & \text{if } \alpha \ne 0, \\ 0, & \text{if } \alpha = 0. \end{cases}$$
Then
   (a) $(\alpha A)^\dagger = \alpha^\dagger A^\dagger$.
   (b) $(\mathrm{diag}(\beta_1, \beta_2, \cdots, \beta_n))^\dagger = \mathrm{diag}(\beta_1^\dagger, \beta_2^\dagger, \cdots, \beta_n^\dagger)$.
10. $(A^\dagger)^* = (A^*)^\dagger$; $(A^\dagger)^\dagger = A$.
11. If A is a nonsingular square matrix, then $A^\dagger = A^{-1}$.
12. If U has orthonormal columns or orthonormal rows, then $U^\dagger = U^*$.
13. If $A = A^*$ and $A = A^2$, then $A^\dagger = A$.
14. $A^\dagger = A^*$ if and only if A^*A is idempotent.
15. If A is normal and k is a positive integer, then $AA^\dagger = A^\dagger A$ and $(A^k)^\dagger = (A^\dagger)^k$.
16. If $U \in \mathbb{C}^{m\times n}$ is of rank n and satisfies $U^\dagger = U^*$, then U has orthonormal columns.
17. If $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary matrices, then $(UAV)^\dagger = V^*A^\dagger U^*$.
18. $A^\dagger = (A^*A)^\dagger A^* = A^*(AA^*)^\dagger$. In particular,
   (a) if $A \in \mathbb{C}^{m\times n}$ ($m \ge n$) has full rank n, then $A^\dagger = (A^*A)^{-1}A^*$;
   (b) if $A \in \mathbb{C}^{m\times n}$ ($m \le n$) has full rank m, then $A^\dagger = A^*(AA^*)^{-1}$.
19. Let $A \in \mathbb{C}^{m\times n}$. Then

   (a) $A^\dagger A$, $AA^\dagger$, $I_n - A^\dagger A$, and $I_m - AA^\dagger$ are orthogonal projections.
   (b) $\mathrm{rank}(A) = \mathrm{rank}(A^\dagger) = \mathrm{rank}(AA^\dagger) = \mathrm{rank}(A^\dagger A)$.
   (c) $\mathrm{rank}(I_n - A^\dagger A) = n - \mathrm{rank}(A)$.
   (d) $\mathrm{rank}(I_m - AA^\dagger) = m - \mathrm{rank}(A)$.
20. $AA^\dagger = \mathrm{Proj}_{\mathrm{range}(A)}$; $A^\dagger A = \mathrm{Proj}_{\mathrm{range}(A^\dagger)}$.
21. Suppose that $A \in F^{m\times n}$, where $F = \mathbb{C}$ or $\mathbb{R}$. Then
   (a) $\mathrm{range}(A) = \mathrm{range}(AA^*) = \mathrm{range}(AA^\dagger)$.
   (b) $\mathrm{range}(A^\dagger) = \mathrm{range}(A^*) = \mathrm{range}(A^*A) = \mathrm{range}(A^\dagger A)$.
   (c) $\ker(A) = \ker(A^*A) = \ker(A^\dagger A)$.
   (d) $\ker(A^\dagger) = \ker(A^*) = \ker(AA^*) = \ker(AA^\dagger)$.
   (e) $\mathrm{range}(A^\dagger A) \oplus \ker(A^\dagger A) = F^n$.
   (f) $\mathrm{range}(AA^\dagger) \oplus \ker(AA^\dagger) = F^m$.
22. If $A = A_1 + A_2 + \cdots + A_k$, $A_i^*A_j = 0$, and $A_iA_j^* = 0$, for all $i, j = 1, \cdots, k$, $i \ne j$, then $A^\dagger = A_1^\dagger + A_2^\dagger + \cdots + A_k^\dagger$.
23. If A is an $m\times r$ matrix of rank r and B is an $r\times n$ matrix of rank r, then $(AB)^\dagger = B^\dagger A^\dagger$.
24. $(A^*A)^\dagger = A^\dagger(A^*)^\dagger$; $(AA^*)^\dagger = (A^*)^\dagger A^\dagger$.
25. [Gre66] Each one of the following conditions is necessary and sufficient for $(AB)^\dagger = B^\dagger A^\dagger$:
   (a) $\mathrm{range}(BB^*A^*) \subseteq \mathrm{range}(A^*)$ and $\mathrm{range}(A^*AB) \subseteq \mathrm{range}(B)$.
   (b) $A^\dagger ABB^*$ and $A^*ABB^\dagger$ are both Hermitian matrices.
   (c) $A^\dagger ABB^*A^* = BB^*A^*$ and $BB^\dagger A^*AB = A^*AB$.
   (d) $A^\dagger ABB^*A^*ABB^\dagger = BB^*A^*A$.
   (e) $A^\dagger AB = B(AB)^\dagger AB$ and $BB^\dagger A^* = A^*AB(AB)^\dagger$.
26. $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$, where $\otimes$ denotes the Kronecker product.
27. $A^\dagger = \lim_{\alpha\to 0} A^*(\alpha I + AA^*)^{-1} = \lim_{\alpha\to 0}(\alpha I + A^*A)^{-1}A^*$.
28. $A^\dagger = \sum_{j=1}^{\infty} A^*(I + AA^*)^{-j} = \sum_{j=1}^{\infty}(I + A^*A)^{-j}A^*$.
29. (Continuity of pseudo-inverse) Suppose that $A \in F^{m\times n}$ and $E \in F^{m\times n}$, where $F = \mathbb{C}$ or $\mathbb{R}$. Then $\lim_{E\to 0}(A+E)^\dagger = A^\dagger$ if and only if there is $\epsilon > 0$ such that $\mathrm{rank}(A+E) = \mathrm{rank(A)}$ when $\|E\|_2 \le \epsilon$.
30. Let $A \in \mathbb{C}^{m\times n}$ be of rank r where $0 < r < \min\{m,n\}$. Suppose that A can be partitioned as
$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$
where $A_{11} \in \mathbb{C}^{r\times r}$ and $\mathrm{rank}(A_{11}) = r$. Then
$$A^\dagger = \begin{bmatrix} A_{11}^*XA_{11}^* & A_{11}^*XA_{21}^* \\ A_{12}^*XA_{11}^* & A_{12}^*XA_{21}^* \end{bmatrix},$$
where
$$X = (A_{11}A_{11}^* + A_{12}A_{12}^*)^{-1}A_{11}(A_{11}^*A_{11} + A_{21}^*A_{21})^{-1}.$$

Applications

- Handbook of Lin. Algebra (20 ✔/ 6 ✔/ 4 ✖) (Bernauer, H., Regensburger '23)

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- Recent results in operator theory

Reverse order law for the Moore–Penrose inverse ☆

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ABSTRACT

In this paper we present new results related to the reverse order law for the Moore–Penrose inverse of operators on Hilbert spaces. Some finite-dimensional results are extended to infinite-dimensional settings.

1. Introduction

In this paper we extend some results from [15] to infinite-dimensional settings. Among other things, we obtain the reverse order law for the Moore–Penrose inverse as a corollary. We use the matrix form of a linear bounded operator, and this matrix form is induced by some natural decompositions of Hilbert spaces.

In the rest of the Introduction we formulate two auxiliary results. In Section 2 we present the results related to the reverse order rule for the Moore–Penrose inverse of Hilbert space operators with closed range. The present paper is the extension of results from [15] to infinite-dimensional settings.

2. Reverse order law

In this section we prove the results concerning the reverse order law for the Moore–Penrose inverse.

Theorem 2.2. *Let X, Y, Z be Hilbert spaces, and let $A \in \mathcal{L}(Y,Z)$, $B \in \mathcal{L}(X,Y)$ be such that A, B, AB have closed ranges. Then the following statements hold:*

(a) $AB(AB)^\dagger = ABB^\dagger A^\dagger \Leftrightarrow A^*AB = BB^\dagger A^*AB \Leftrightarrow \mathcal{R}(A^*AB) \subseteq \mathcal{R}(B) \Leftrightarrow B^\dagger A^\dagger \in (AB)\{1,2,3\}$;
(b) $(AB)^\dagger AB = B^\dagger A^\dagger AB \Leftrightarrow ABB^* = ABB^*A^\dagger A \Leftrightarrow \mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*) \Leftrightarrow B^\dagger A^\dagger \in (AB)\{1,2,4\}$;
(c) *The following statements are equivalent:*
  (1) $(AB)^\dagger = B^\dagger A^\dagger$;
  (2) $AB(AB)^\dagger = ABB^\dagger A^\dagger$ *and* $(AB)^\dagger AB = B^\dagger A^\dagger AB$;
  (3) $A^*AB = BB^\dagger A^*AB$ *and* $ABB^* = ABB^*A^\dagger A$;
  (4) $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ *and* $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$.

Proof. The operators A and B have the same matrix representations as in the previous theorem. The following products will be useful:

$$AB = \begin{bmatrix} A_1B_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad (AB)^\dagger = \begin{bmatrix} (A_1B_1)^\dagger & 0 \\ 0 & 0 \end{bmatrix}, \qquad B^\dagger A^\dagger = \begin{bmatrix} B_1^{-1}A_1^*D^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

First, we find the equivalent expressions for our statements in terms of A_1, A_2 and B_1.

(a) 1. $AB(AB)^\dagger = ABB^\dagger A^\dagger \Leftrightarrow A_1B_1(A_1B_1)^\dagger = A_1A_1^*D^{-1}$. Here $A_1B_1(A_1B_1)^\dagger$ is Hermitian, so $[A_1A_1^*, D^{-1}] = 0$.
2. $A^*AB = BB^\dagger A^*AB \Leftrightarrow A_2^*A_1 = 0$.
3. Notice that $\mathcal{R}(A^*AB) \subset \mathcal{R}(B)$ if and only if $BB^\dagger A^*AB = A^*AB$, so $2 \Leftrightarrow 3$.
4. If we check properly the Penrose equations, then we see that: $B^\dagger A^\dagger \in (AB)\{1,2,3\} \Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1$ and $[A_1A_1^*, D^{-1}] = 0$.

Now, we prove the following: $1 \Leftrightarrow 2$, $4 \Rightarrow 2$ and $1 \Rightarrow 4$.
We prove $1 \Leftrightarrow 2$. Notice that

$$A_1B_1(A_1B_1)^\dagger = A_1A_1^*D^{-1} \quad \Leftrightarrow \quad (A_1B_1)^\dagger = (A_1B_1)^\dagger A_1A_1^*D^{-1}.$$

The last statement is obtained by multiplying the first expression by $(A_1B_1)^\dagger$ from the left side, or multiplying the second expression by A_1B_1 from the left side, and using $A_1A_1^* = A_1B_1B_1^{-1}A_1^*$. Now, there is a chain of the equivalences:

$$\begin{aligned} (A_1B_1)^\dagger = (A_1B_1)^\dagger A_1A_1^*D^{-1} \quad &\Leftrightarrow \quad (A_1B_1)^\dagger\big(A_1A_1^* + A_2A_2^*\big) = (A_1B_1)^\dagger A_1A_1^* \\ &\Leftrightarrow \quad (A_1B_1)^\dagger A_2A_2^* = 0 \quad \Leftrightarrow \quad \mathcal{R}\big(A_2A_2^*\big) \subset \mathcal{N}\big((A_1B_1)^\dagger\big) \\ &\Leftrightarrow \quad \mathcal{R}(A_2) \subset \mathcal{N}\big((A_1B_1)^*\big) \quad \Leftrightarrow \quad B_1^*A_1^*A_2 = 0 \quad \Leftrightarrow \quad A_1^*A_2 = 0. \end{aligned}$$

Therefore, we have just proved that $1 \Leftrightarrow 2$.

Now we prove $1 \Rightarrow 4$. If we multiply $A_1B_1(A_1B_1)^\dagger = A_1A_1^*D^{-1}$ by A_1B_1 from the right side, we get $A_1A_1^*D^{-1}A_1 = A_1$. Thus, 4 holds.

Finally, we prove $4 \Rightarrow 2$. If $A_1A_1^*D^{-1}A_1 = A_1$ and $[A_1A_1^*, D^{-1}] = 0$, then $A_1A_1^*A_1 = DA_1 = A_1A_1^*A_1 + A_2A_2^*A_1$, implying that $A_2A_2^*A_1 = 0$. Hence, $\mathcal{R}(A_1) \subset \mathcal{N}(A_2A_2^*) = \mathcal{N}(A_2^*)$, so $A_2^*A_1 = 0$. Thus, 2 holds.

Notice that the equivalence $3 \Leftrightarrow 4$ is proved in [8], also.

(b) 1. $(AB)^\dagger AB = B^\dagger A^\dagger AB \Leftrightarrow (A_1B_1)^\dagger A_1B_1 = B_1^{-1}A_1^*D^{-1}A_1B_1$. Moreover, $(A_1B_1)^\dagger A_1B_1$ is Hermitian, so $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$.
2. $ABB^* = ABB^*A^\dagger A \Leftrightarrow A_1B_1B_1^*A_1^*D^{-1}A_1 = A_1B_1B_1^*$ and $A_1B_1B_1^*A_1^*D^{-1}A_2 = 0$.
3. Notice that $\mathcal{R}(BB^*A^*) \subset \mathcal{R}(A^*)$ if and only if $A^\dagger ABB^*A^* = BB^*A^*$, which is equivalent to $ABB^*A^\dagger A = ABB^*$. Hence, $2 \Leftrightarrow 3$.
4. The Penrose equations imply that: $B^\dagger A^\dagger \in (AB)\{1,2,4\} \Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1$ and $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$.

We prove $1 \Rightarrow 4 \Rightarrow 2 \Rightarrow 1$.

Suppose that 1 holds. If we multiply $(A_1B_1)^\dagger A_1B_1 = B_1^{-1}A_1^*D^{-1}A_1B_1$ by A_1B_1 from the left side, we obtain $A_1 = A_1A_1^*D^{-1}A_1$. Furthermore, $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$ holds. Therefore, $1 \Rightarrow 4$.

Suppose that 4 holds. Obviously, $A_1B_1B_1^*A_1^*D^{-1}A_1 = A_1A_1^*D^{-1}A_1B_1B_1^* = A_1B_1B_1^*$. Thus, the first equality of 2 holds. The second equality of 2 also holds, since $A_1^*D^{-1}A_2 = 0 \Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1$, which is shown in the proof of Theorem 2.1. Here we use again $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$. Consequently, $4 \Rightarrow 2$.

In order to prove that $2 \Rightarrow 1$, we multiply $A_1B_1B_1^*A_1^*D^{-1}A_1 = A_1B_1B_1^*$ by $(A_1B_1)^\dagger$ from the left side. It follows that $B_1^*A_1^*D^{-1}A_1 = (A_1B_1)^\dagger A_1B_1B_1^*$, so $(A_1B_1)^\dagger A_1B_1 = B_1^*A_1^*D^{-1}A_1(B_1^*)^{-1}$ which is equivalent to $(A_1B_1)^\dagger A_1B_1 = B_1^{-1}A_1^*D^{-1}A_1B_1$. Hence, $2 \Rightarrow 1$.

Notice that $3 \Leftrightarrow 4$ is also proved in [8].

Finally, the part (c) follows from the parts (a) and (b). □

We also prove the following result.

Theorem 2.3. *Let X, Y, Z be Hilbert spaces, and let $A \in \mathcal{L}(Y,Z)$, $B \in \mathcal{L}(X,Y)$ be such that A, B, AB have closed ranges. Then we have:*

(a) $AB(AB)^\dagger A = ABB^\dagger \Leftrightarrow A^*ABB^\dagger = BB^\dagger A^*A \Leftrightarrow \mathcal{R}(A^*AB) \subseteq \mathcal{R}(B) \Leftrightarrow B^\dagger A^\dagger \in (AB)\{1,2,3\}$;
(b) $B(AB)^\dagger AB = A^\dagger AB \Leftrightarrow A^\dagger ABB^* = BB^*A^\dagger A \Leftrightarrow \mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*) \Leftrightarrow B^\dagger A^\dagger \in (AB)\{1,2,4\}$;
(c) *The following three statements are equivalent:*
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  (2) $AB(AB)^\dagger A = ABB^\dagger$ *and* $B(AB)^\dagger AB = A^\dagger AB$;
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Proof. The operators A and B have the same matrix representations as in the previous theorem. First, we find equivalent expressions, in the terms of A_1, A_2 and B_1, for our assumptions.

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3. Notice that $\mathcal{R}(A^*AB) \subset \mathcal{R}(B)$ if and only if $BB^\dagger A^*AB = A^*AB$, so $2 \Leftrightarrow 3$.
4. If we check properly the Penrose equations, then we see that: $B^\dagger A^\dagger \in (AB)\{1,2,3\} \Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1$ and $[A_1A_1^*, D^{-1}] = 0$.

Now, we prove the following: $1 \Leftrightarrow 2$, $4 \Rightarrow 2$ and $1 \Rightarrow 4$.
We prove $1 \Leftrightarrow 2$. Notice that

$$A_1B_1(A_1B_1)^\dagger = A_1A_1^*D^{-1} \quad \Leftrightarrow \quad (A_1B_1)^\dagger = (A_1B_1)^\dagger A_1A_1^*D^{-1}.$$

The last statement is obtained by multiplying the first expression by $(A_1B_1)^\dagger$ from the left side, or multiplying the second expression by A_1B_1 from the left side, and using $A_1A_1^* = A_1B_1B_1^{-1}A_1^*$. Now, there is a chain of the equivalences:

$$\begin{aligned}(A_1B_1)^\dagger = (A_1B_1)^\dagger A_1A_1^*D^{-1} \quad &\Leftrightarrow \quad (A_1B_1)^\dagger\big(A_1A_1^* + A_2A_2^*\big) = (A_1B_1)^\dagger A_1A_1^* \\ &\Leftrightarrow \quad (A_1B_1)^\dagger A_2A_2^* = 0 \quad \Leftrightarrow \quad \mathcal{R}\big(A_2A_2^*\big) \subset \mathcal{N}\big((A_1B_1)^\dagger\big) \\ &\Leftrightarrow \quad \mathcal{R}(A_2) \subset \mathcal{N}\big((A_1B_1)^*\big) \quad \Leftrightarrow \quad B_1^*A_1^*A_2 = 0 \quad \Leftrightarrow \quad A_1^*A_2 = 0.\end{aligned}$$

Therefore, we have just proved that $1 \Leftrightarrow 2$.
Now we prove $1 \Rightarrow 4$. If we multiply $A_1B_1(A_1B_1)^\dagger = A_1A_1^*D^{-1}$ by A_1B_1 from the right side, we get $A_1A_1^*D^{-1}A_1 = A_1$. Thus, 4 holds.
Finally, we prove $4 \Rightarrow 2$. If $A_1A_1^*D^{-1}A_1 = A_1$ and $[A_1A_1^*, D^{-1}] = 0$, then $A_1A_1^*A_1 = DA_1 = A_1A_1^*A_1 + A_2A_2^*A_1$, implying that $A_2A_2^*A_1 = 0$. Hence, $\mathcal{R}(A_1) \subset \mathcal{N}(A_2A_2^*) = \mathcal{N}(A_2^*)$, so $A_2^*A_1 = 0$. Thus, 2 holds.
Notice that the equivalence $3 \Leftrightarrow 4$ is proved in [8], also.

(b) 1. $(AB)^\dagger AB = B^\dagger A^\dagger AB \Leftrightarrow (A_1B_1)^\dagger A_1B_1 = B_1^{-1}A_1^*D^{-1}A_1B_1$. Moreover, $(A_1B_1)^\dagger A_1B_1$ is Hermitian, so $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$.
2. $ABB^* = ABB^*A^\dagger A \Leftrightarrow A_1B_1B_1^*A_1^*D^{-1}A_1 = A_1B_1B_1^*$ and $A_1B_1B_1^*A_1^*D^{-1}A_2 = 0$.
3. Notice that $\mathcal{R}(BB^*A^*) \subset \mathcal{R}(A^*)$ if and only if $A^\dagger ABB^*A^* = BB^*A^*$, which is equivalent to $ABB^*A^\dagger A = ABB^*$. Hence, $2 \Leftrightarrow 3$.
4. The Penrose equations imply that: $B^\dagger A^\dagger \in (AB)\{1,2,4\} \Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1$ and $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$.

We prove $1 \Rightarrow 4 \Rightarrow 2 \Rightarrow 1$.
Suppose that 1 holds. If we multiply $(A_1B_1)^\dagger A_1B_1 = B_1^{-1}A_1^*D^{-1}A_1B_1$ by A_1B_1 from the left side, we obtain $A_1 = A_1A_1^*D^{-1}A_1$. Furthermore, $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$ holds. Therefore, $1 \Rightarrow 4$.
Suppose that 4 holds. Obviously, $A_1B_1B_1^*A_1^*D^{-1}A_1 = A_1A_1^*D^{-1}A_1B_1B_1^* = A_1B_1B_1^*$. Thus, the first equality of 2 holds. The second equality of 2 also holds, since $A_1^*D^{-1}A_2 = 0 \Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1$, which is shown in the proof of Theorem 2.1. Here we use again $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$. Consequently, $4 \Rightarrow 2$.
In order to prove that $2 \Rightarrow 1$, we multiply $A_1B_1B_1^*A_1^*D^{-1}A_1 = A_1B_1B_1^*$ by $(A_1B_1)^\dagger$ from the left side. It follows that $B_1^*A_1^*D^{-1}A_1 = (A_1B_1)^\dagger A_1B_1B_1^*$, so $(A_1B_1)^\dagger A_1B_1 = B_1^*A_1^*D^{-1}A_1(B_1^*)^{-1}$ which is equivalent to $(A_1B_1)^\dagger A_1B_1 = B_1^{-1}A_1^*D^{-1}A_1B_1$. Hence, $2 \Rightarrow 1$.
Notice that $3 \Leftrightarrow 4$ is also proved in [8].

Finally, the part (c) follows from the parts (a) and (b). □

We also prove the following result.

Theorem 2.3. *Let X, Y, Z be Hilbert spaces, and let $A \in \mathcal{L}(Y,Z)$, $B \in \mathcal{L}(X,Y)$ be such that A, B, AB have closed ranges. Then we have:*

(a) $AB(AB)^\dagger A = ABB^\dagger \Leftrightarrow A^*ABB^\dagger = BB^\dagger A^*A \Leftrightarrow \mathcal{R}(A^*AB) \subseteq \mathcal{R}(B) \Leftrightarrow B^\dagger A^\dagger \in (AB)\{1,2,3\}$;
(b) $B(AB)^\dagger AB = A^\dagger AB \Leftrightarrow A^\dagger ABB^* = BB^*A^\dagger A \Leftrightarrow \mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*) \Leftrightarrow B^\dagger A^\dagger \in (AB)\{1,2,4\}$;
(c) *The following three statements are equivalent:*
  (1) $(AB)^\dagger = B^\dagger A^\dagger$;
  (2) $AB(AB)^\dagger A = ABB^\dagger$ *and* $B(AB)^\dagger AB = A^\dagger AB$;
  (3) $A^*ABB^\dagger = BB^\dagger A^*A$ *and* $A^\dagger ABB^* = BB^*A^\dagger A$.

Proof. The operators A and B have the same matrix representations as in the previous theorem. First, we find equivalent expressions, in the terms of A_1, A_2 and B_1, for our assumptions.

Applications

- Handbook of Lin. Algebra (20 ✔/ 6 ✔/ 4 ✖) (Bernauer, H., Regensburger '23)
  - each proof takes < 1 second
  - proofs consist of up to 226 polynomials
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- Homological algebra

$$\begin{tikzcd}
 & \ker(a) \arrow[r] \arrow[d] & \ker(b) \arrow[r] \arrow[d] & \ker(c) \arrow[d] \arrow[dddll, rounded corners] & \\
 & A \arrow[r, "f"'] \arrow[d, "a"] & B \arrow[r, "g"'] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"'] \arrow[d] & B' \arrow[r, "g'"'] \arrow[d] & C' \arrow[d] & \\
 & \operatorname{coker}(a) \arrow[r] & \operatorname{coker}(b) \arrow[r] & \operatorname{coker}(c) &
\end{tikzcd}$$

Computing short proofs

Theorem (Djordjević, Dinčić '09) A, B matrices such that AB exists.

$$B^\dagger (ABB^\dagger)^\dagger = (A^\dagger AB)^\dagger A^\dagger = B^\dagger A^\dagger \quad \Rightarrow \quad (AB)^\dagger = B^\dagger A^\dagger$$

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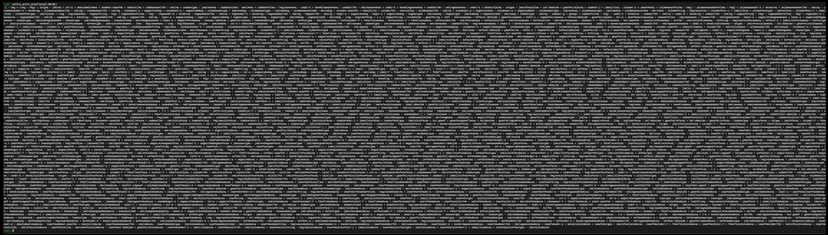

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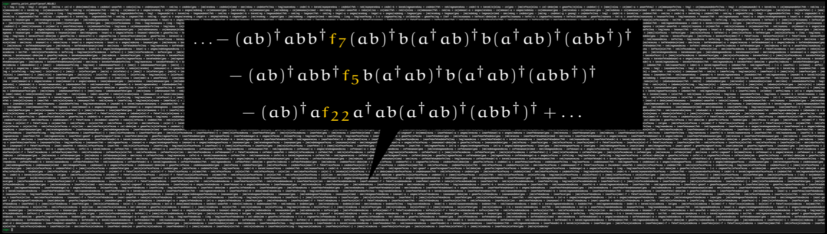

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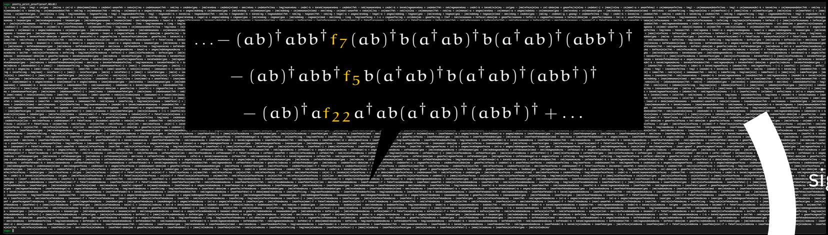

$$\ldots - (ab)^\dagger abb^\dagger f_7 (ab)^\dagger b (a^\dagger ab)^\dagger b (a^\dagger ab)^\dagger (abb^\dagger)^\dagger$$
$$- (ab)^\dagger abb^\dagger f_5 b (a^\dagger ab)^\dagger b (a^\dagger ab)^\dagger (abb^\dagger)^\dagger$$
$$- (ab)^\dagger a f_{22} a^\dagger ab (a^\dagger ab)^\dagger (abb^\dagger)^\dagger + \ldots$$

sig-GB
+
LP

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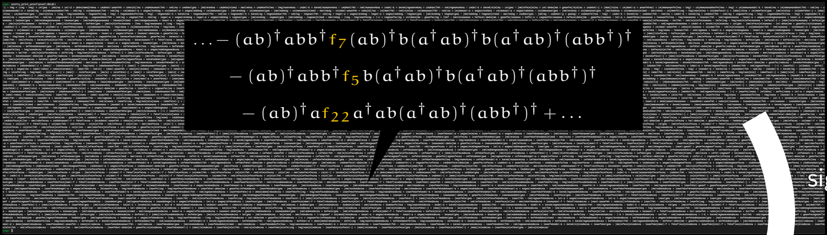

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$$- (ab)^\dagger a f_{22} a^\dagger ab(a^\dagger ab)^\dagger (abb^\dagger)^\dagger + \dots$$

sig-GB + LP

Another proof

$$\begin{aligned}(ab)^\dagger - b^\dagger a^\dagger = {} & f_{21} - f_{10} + b^\dagger f_{14} - f_{12}(ab)^\dagger - b^\dagger(abb^\dagger)^\dagger f_{11} + b^\dagger(abb^\dagger)^\dagger f_{15} \\ & + (a^\dagger ab)^\dagger a^\dagger f_9 (ab)^\dagger - b^* f_{23}((ab)^\dagger)^*(ab)^\dagger - f_{21} ab(ab)^\dagger + f_{22} ab(ab)^\dagger \\ & - f_{39}(a^\dagger)^*((ab)^\dagger)^*(ab)^\dagger + b^\dagger(abb^\dagger)^\dagger((abb^\dagger)^\dagger)^*(b^\dagger)^* f_{31} - b^\dagger f_{14} d^* b^* (a^\dagger)^* \\ & + (a^\dagger ab)^\dagger a^\dagger ab f_{12}(ab)^\dagger - b^\dagger(abb^\dagger)^\dagger f_{15}((ab)^\dagger)^* b^* (a^\dagger)^* \\ & + f_{20} b^*(a^\dagger)^*((ab)^\dagger)^*(ab)^\dagger + (a^\dagger ab)^\dagger a^\dagger abb^* f_{23}((ab)^\dagger)^*(ab)^\dagger\end{aligned}$$

What about false statements?

$$\forall A, B, C \ : \ (A \neq 0 \ \wedge \ AB = AC) \ \Rightarrow \ B = C$$

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Idea: make ansatz
with matrices
of fixed size

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \qquad C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

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SAT
+
Hensel lifting

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

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Does this always work? – No.

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Does this always work? – No.

Will a better algorithm always work? – No.

Does this work often enough?

What about false statements?

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Does this always work? – No.

Will a better algorithm always work? – No.

Does this work often enough? – Don't know yet.

Conclusion

Summary

- Framework for proving first-order statements about linear operators
- Approach yields semi-decision procedure
- We can find minimal assumptions, short proofs, counterexamples,. . .

Outlook

- Use state-of-the-art techniques from theorem proving
- Include operator series, analytic properties, uncertainty,. . .
- Further applications

F

Hi chatGPT

Hello! How can I assist you today?

F

Imagine you are a mathematician and you want to prove a statement about linear operators. What do you do?

I would use computer algebra.

Regenerate response

Send a message